
A TIGHT BOUND ON THE MAXIMUM INTERFERENCE OF RANDOM SENSORS IN THE HIGHWAY MODEL

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ABSTRACT. Consider n sensors whose positions are represented by n uniform, independent and identically distributed random variables assuming values in the open unit interval $(0, 1)$. A natural way to guarantee connectivity in the resulting sensor network is to assign to each sensor as its range, the maximum of the two possible distances to its two neighbors. The interference at a given sensor is defined as the number of sensors that have this sensor within their range. In this paper we prove that the expected maximum interference of the sensors is $\Theta(\sqrt{\ln n})$.

1 Introduction

The broadcast nature of wireless communication implies that interference with other transmissions is inevitable. Interference can be caused by sources inside or outside the system and comes in many forms. Co-channel interference is caused by other wireless devices transmitting on the same frequency. Such interference can make it impossible for a receiver to decode a transmission unless the signal power of the intended source is significantly higher than the combined strength of the signal received from the interfering sensors. Wireless devices are designed to admit a certain maximum level of interference. It is therefore crucial to understand the maximum possible interference that may be experienced by any element in a wireless network.

In this paper we study the expected maximum interference for n sensors placed at random in the *highway model*. According to this model, n sensors are represented by n uniform, independent and identically distributed random variables in the open unit interval $(0, 1)$.

Since only nodes whose transmissions can reach a node can cause interference at it, an important way to manage interference is by the use of topology control algorithms. In

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particular, one can assign transmission ranges to nodes with the objective of minimizing interference. On the other hand, the assignment of transmission ranges should also ensure that the network is connected. In the highway model, a natural algorithm is to assign as transmission range to a sensor the maximum distance between its two immediate (from the left and right) neighbors since this is the minimum range required to attain connectivity. We are interested in studying the resulting *interference* among the n sensors. Intuitively, the interference for each sensor i is defined as the number of sensors that have i within their range.

Several papers study interference and network performance degradation. Gupta and Kumar [5] considers the throughput of wireless networks under two models of interference: one is a protocol model that assumes interference to be an all-or-nothing phenomenon and the other a physical model that considers the impact of interfering transmissions on the signal-to-noise ratio. Motivated by this, Jain *et al.* [7] defined the concept of *conflict graph* (a graph indicating which groups of nodes interfere and hence cannot be active at the same time) and study what is the maximum throughput that can be supported by a wireless network given a specific placement of wireless nodes in physical space and a specific traffic workload.

Burkhart *et al.* [3] proposes connectivity preserving and spanner constructions which are interference optimal. [9] considers the *average interference* problem while maintaining connectivity. Closely related to our study is the following problem first proposed in [8]:

Given n nodes in the plane. Connect the nodes by a spanning tree. For each node v we construct a disk centering at v with radius equal to the distance to v 's furthest neighbor in the spanning tree. The interference of a node v is then defined as the number of disks that include node v (not counting the disk of v itself). Find a spanning tree that minimizes the maximum interference.

Choosing transmission radii which minimize the maximum interference while maintaining a connected symmetric communication graph is shown by [2] to be NP-complete. In addition, [6] gives an algorithm which yields a maximum interference in $O(\sqrt{n})$ for any set of n sensors in the plane. For the case of points on a line (i.e., the highway model) [10] shows that if nodes are distributed as an exponential node chain, the algorithm described above for assigning ranges to sensors has maximum interference $\Omega(n)$. They proceed to give an $n^{1/4}$ -approximation algorithm for the problem of finding an assignment of ranges that minimizes interference.

[1] shows that for broadcasting (one-to-all), gossiping (all-to-all), and symmetric gossiping (symmetric all-to-all) the problem of minimizing the maximum interference experienced by any node in the network is hard to approximate within better than a logarithmic factor, unless NP admits slightly superpolynomial time algorithms. They also prove that any approximation algorithm for the problem of minimizing the total transmission power assigned to the nodes in order to guarantee any of the above communication patterns, can be transformed, by maintaining the same performance ratio, into an approximation algorithm for the problem of minimizing the total interference experienced by all the nodes in the network.

Here we study a model where sensors are represented by n uniform, independent and identically distributed random variables in the open unit interval $(0, 1)$. We assign to each sensor as range the maximum of the two possible distances with its two neighbors. For this case, we show a tight bound on the expected maximum interference experienced by any sensor. In particular, Theorem 1 shows that the expected maximum interference is $\Theta(\sqrt{\ln n})$, with high probability. This is in contrast to the result of [10] that the maximum interference for n sensors distributed on a line and connected in the same manner is $\Omega(n)$ in the worst case.

2 Expected Maximum Interference

Let $S = \{x_1, \dots, x_n\}$ be a set of values chosen independently and uniformly at random from the real interval $[0, 1]$ and reordered so that $x_1 < \dots < x_n$. For each $i \in \{2, \dots, n-1\}$, define the *broadcast range*

$$R_i = \max\{x_i - x_{i-1}, x_{i+1} - x_i\}$$

and the *broadcast interval*

$$I_i = [x_i - R_i, x_i + R_i] .$$

For $i = 1$ ($i = n$) define $R_1 = x_2 - x_1$ ($R_n = x_n - x_{n-1}$, respectively) and $I_1 = [x_1 - R_1, x_1 + R_1]$ ($I_n = [x_n - R_n, x_n + R_n]$, respectively).

The *interference at x_i* is then given by

$$Z_i = |\{j \in \{1, \dots, n\} \setminus \{i\} : x_i \in I_j\}| .$$

The *maximum interference in S* , is given by

$$Z_S = \max\{Z_i : i \in \{1, \dots, n\}\} .$$

In this section we prove our main result:

Theorem 1. *With probability $1 - o(1)$, the maximum interference $Z_S \in \Theta(\sqrt{\log n})$.*

This result is an immediate consequence of Lemmas 2 and 4. Throughout this section, we will make use of the relationship between uniformly distributed point sets and exponential random variables [4][Chapter V, Theorem 2.2]. Suppose S is a set of n points independently and uniformly distributed in $[0, 1]$ whose elements are x_1, \dots, x_n in sorted order. Let X_0, \dots, X_n be Exponential(1) random variables, let $x'_i = \sum_{j=0}^{i-1} X_j$, and let $x''_i = x'_i / x'_{n+1}$. Then x''_1, \dots, x''_n have the same distribution as x_1, \dots, x_n .

Because of the above relationship we will, throughout this section, use the convention that X_0, \dots, X_n are Exponential(1) random variables, $x_i = \sum_{j=0}^{i-1} X_j$, and $S = \{x_1, \dots, x_n\}$. This definition of S , x_1, \dots, x_n , and X_0, \dots, X_n will be implicit in the statements of all subsequent results and in all proofs.

2.1 The Lower Bound

We prove our lower-bound by defining a configuration of points that leads to an element with interference $\Omega(\sqrt{\log n})$ and then showing that, with high probability, this configuration occurs somewhere in our point set.

A sequence of numbers X_0, \dots, X_k forms a *k-frame* if

$$1 \leq X_0 \leq 2$$

and

$$X_{i-1}/4 \leq X_i \leq X_{i-1}/2 \text{ ,}$$

for all $i \in \{1, \dots, k\}$. Notice that, if X_0, \dots, X_k form a *k-frame*, then x_{k+1} is a node that has interference at least k . The next lemma shows that this situation is not too unlikely:

Lemma 1. *If X_0, \dots, X_k are a sequence of independent Exponential(1) random variables, then the probability that X_0, \dots, X_k form a *k-frame* is at least $2^{-(k+2)^2}$.*

Proof. Recall that an Exponential(1) random variable X has cumulative distribution function

$$\Pr\{X \leq x\} = 1 - e^{-x} \text{ .}$$

Next, observe that, in a frame,

$$4^{-i} \leq X_i \leq 2^{-i} \text{ ,}$$

for all $i \in \{0, \dots, k\}$. Let $F(X)$ be the event “ X is a frame.” Then,

$$\begin{aligned} \Pr\{F(X_0, \dots, X_{i+1}) \mid F(X_0, \dots, X_i)\} &= \Pr\{X_{i+1} \in [X_i/4, X_i/2] \mid F(X_0, \dots, X_i)\} \\ &\geq \Pr\{X_{i+1} \in [4^{-i}/4, 4^{-i}/2]\} \\ &= \Pr\{X_{i+1} \in [2^{-(2i+2)}, 2^{-(2i+1)}]\} \\ &= \exp(-2^{-(2i+2)}) - \exp(2^{-(2i+1)}) \\ &\geq 2^{-(2i+3)} \text{ ,} \end{aligned}$$

where the last inequality holds for all $i \geq 0$. Therefore,

$$\begin{aligned}
\Pr\{F(X_0, \dots, X_k)\} &= \Pr\{X_0 \in [1, 2]\} \cdot \prod_{i=1}^k \Pr\{F(X_0, \dots, X_i) \mid F(X_0, \dots, X_{i-1})\} \\
&= e^{-1}(1 - e^{-1}) \cdot \prod_{i=1}^k \Pr\{F(X_0, \dots, X_i) \mid F(X_0, \dots, X_{i-1})\} \\
&= e^{-1}(1 - e^{-1}) \cdot \prod_{i=1}^k \Pr\{X_i \in [X_{i-1}/4, X_{i-1}/2]\} \mid F(X_0, \dots, X_{i-1}) \\
&\geq e^{-1}(1 - e^{-1}) \cdot \prod_{i=1}^k \Pr\{X_i \in [4^{-i}/4, 4^{-i}/2]\} \\
&\geq e^{-1}(1 - e^{-1}) \cdot \prod_{i=1}^k 2^{-(2i+1)} \\
&= e^{-1}(1 - e^{-1}) \cdot 2^{-\sum_{i=1}^k (2i+1)} \\
&= e^{-1}(1 - e^{-1}) \cdot 2^{-(k^2+2k)} \\
&\geq 2^{-(k+2)^2}
\end{aligned}$$

as required. \square

Lemma 2 (Lower Bound). *With probability at least $1 - \exp(-n^{1-c}/\sqrt{c \log n})$, there exists some element of S that has interference at least $\lfloor \sqrt{c \log n} \rfloor - 2$.*

Proof. Let $k = \lfloor \sqrt{c \log n} \rfloor - 2$. By Lemma 1, $X_{jk}, \dots, X_{j(k+k)}$ have probability at least $2^{-(k+2)^2} = n^{-c}$ of forming a k -frame, in which case $x_{j(k+k)}$ has interference at least k . Since this is true, independently, for any $j \in \{0, \dots, \lfloor n/k \rfloor\}$, the probability that there is no element of S with interference greater than k is at most

$$(1 - n^{-c})^{\lfloor n/k \rfloor} \leq \exp(-\lfloor n^{1-c}/k \rfloor),$$

as required. \square

2.2 The Upper Bound

We begin our upper-bound proof by studying a variant of interference that is 1-sided and that considers only interference generated by transmitters that are nearby. The *left-interference* of an element $x_t \in S$ is the number of elements $x_i \in S$ such that $x_i < x_t$ and $x_t - x_i \leq \max\{x_i - x_{i-1}, x_{i+1} - x_i\}$. The *short-range left-interference* of x_t is defined in the same way, except only counting those elements x_i such that $X_{i-1} \leq 1$. (Note that this implies $x_t - x_i \leq 1$.)

Lemma 3. *The maximum short-range left-interference of any element in S is at most $\sqrt{c \log n}$ with probability at least $1 - n^{-\Omega(c)}$.*

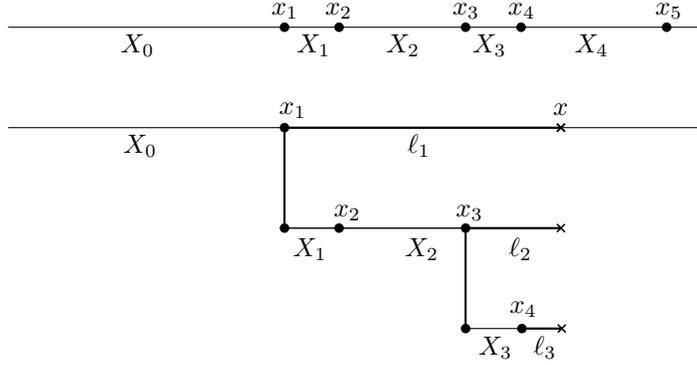


Figure 1: A process that leads to an interference of 3 at x . The process ends because $X_4 > \ell_3$.

Proof. We will actually prove something stronger, namely that the short-range left-interference of any point $x \in \mathbb{R}$ is at most $\sqrt{c \log n}$ with probability at least $1 - n^{-\Omega(c)}$. We first observe that the maximum value of the short range interference occurs when x is of the form $x_i + X_{i-1}$, for some $i \in \{1, \dots, n\}$ where $X_{i-1} \leq 1$.

Consider the following process, that begins with X_0 and upper-bounds the short-range left-interference at $x = x_1 + X_0 = 2X_0$ (see Figure 1). If $X_0 > 1$, the process immediately ends. Otherwise, the process proceeds in rounds where, in round i , there is a length ℓ_i . Initially $\ell_i = X_0$. During round i , we generate $X_{r_{i-1}+1}, \dots, X_{r_i}$ until $\sum_{j=1}^r X_{r_{i-1}+j} \geq \ell_i/2$. If $\sum_{j=1}^r X_j \geq \ell_i$, then the process ends. Otherwise, we set $\ell_{i+1} = \ell_i - \sum_{j=1}^r X_j$ and continue onto round $i + 1$.

Notice that, in this process, the only elements that might contribute to the short-range left-interference at x are x_1 and those x_i where X_{i-1} completes a round other than the final round. Thus, if the above process terminates during round k , then the short-range left-interference at x is at most k .

Now, observe that in round i , $\ell_i \leq 1/2^{i-1}$. Therefore, the probability of continuing to round $i + 1$ from round i is at most

$$\Pr\{X_{r_{i-1}+1} \leq 1/2^{i-1}\} = 1 - e^{-2^{-i+1}} \leq 2^{-i+1}.$$

Therefore, the probability of continuing up to round k is at most

$$\prod_{i=1}^{k-1} 2^{-i+1} = 2^{-\sum_{i=1}^{k-1} (i+1)} = 2^{-(k+2)(k-1)/2} \leq 2^{-k^2/2},$$

for $k \geq 2$. Taking $k = \sqrt{c \log n}$, we find that this probability is at most $1/n^{c/2}$. Therefore, the probability that there is *any* point $x \in \mathbb{R}$ with short-range left-interference greater than $\sqrt{c \log n}$ is at most $1/n^{c/2-1}$, as required. \square

Finally, we have all the pieces needed to complete the upper bound:

Lemma 4 (Upper Bound). *With probability at least $1 - n^{-\Omega(c)}$, the maximum interference of any element in S is at most $\sqrt{c \log n}$.*

Proof. We consider only left-interference, since the right-interference can be bounded in a symmetric way. Consider some element x_t . The left-interference of x_t is generated by some elements x_{i_0}, \dots, x_{i_k} where $x_{i_k} < \dots < x_{i_0} < x_t$. Lemma 3 already bounds the number of elements of this sequence where $X_{i_{j-1}} \leq 1$. Thus, all that remains is to bound the number of elements x_{i_j} where $X_{i_j} > 1$.

Observe, as in the proof of Lemma 3, that, for any $j \in \{1, \dots, k\}$, in order for x_{i_j} to interfere with x_t we must have

$$X_{i_{j-1}} \geq x_t - x_{i_j} ,$$

which implies that $X_{i_{j-1}} \geq 2X_{i_{j-1}-1}$ for all $j \in \{1, \dots, k\}$. Therefore, if we have 2^r elements with $X_{i_j} > 1$, then we have some element $X_{i_{k-1}} > 2^r$. The probability that a particular X_i is greater than 2^r is e^{-2^r} . Therefore, the probability that there exists any X_i greater than 2^r is at most ne^{-2^r} . Setting $r = \log(d \ln n)$ for a sufficiently large constant $d > 1$ makes this probability at most n^{1-d} , and completes the proof. \square

3 Conclusion

In this paper we have investigated the receiver interference for a set of random sensors on a line (also known as the highway model) and proved a tight bound on the value of the expected maximum interference. An interesting question would be to look at probability distributions other than uniform for the arrangement of sensors. Also, bounds for the case of randomly distributed sensors in two dimensions would be interesting. As in the one-dimensional case studied here where the range of the sensors is assigned as the maximum distance to their two neighbors, an analysis of the two-dimensional case must be preceded by an assignment of sensor ranges. A natural choice would be assign each sensor a range equal the maximum distance to its neighbors in the minimum spanning tree of the point set.

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