

# Local 7-Coloring for Planar Subgraphs of Unit Disk Graphs

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**Abstract.** The problem of computing locally a coloring of an arbitrary planar subgraph of a unit disk graph is studied. Each vertex knows its coordinates in the plane, can directly communicate with all its neighbors within unit distance. Using this setting, first a simple algorithm is given whereby each vertex can compute its color in a 9-coloring of the planar graph using only information on the subgraph located within at most 9 hops away from it in the original unit disk graph. A more complicated algorithm is then presented whereby each vertex can compute its color in a 7-coloring of the planar graph using only information on the subgraph located within a constant number of hops away from it.

## 1 Introduction

Graph vertex coloring is the problem of assigning colors to the vertices of a given graph so that no two adjacent vertices share the same color. The problem enjoys many practical and theoretical applications i.e. in scheduling, register allocation in compilers, and pattern matching and many others. Because of its relation to frequency assignment in mobile and wireless networks, the vertex coloring problem has been considered in many research papers, (e.g., see the survey papers Galinier et al. [7], Tuza [18] and the source book [10] of coloring problems by Jensen and Toft).

In this paper we are interested in the graph vertex coloring as applicable to wireless ad-hoc networks. The wireless ad-hoc networks of interest to us are geometrically embedded in the plane and consist of a number of nodes, say  $n$ , whereby two nodes are adjacent if and only if they are within the transmission range of each other. If all the nodes have the same transmission range then these networks are known as *unit disk graphs*. For such graphs there have been several papers in the literature addressing the coloring problem. Among these it is worth mentioning the work by Marathe et al. [13] (presenting an on-line coloring heuristic which achieves a competitive ratio of 6 for unit disk graphs: the heuristic does not need a geometric representation of unit disk graphs which is used only in establishing the performance guarantees of the heuristics), Graf et al. [9] (which improves on a result of Clark, Colbourn and Johnson (1990) and shows that the coloring problem for unit disk graphs remains NP-complete for any fixed number of colors  $k \geq 3$ ), Caragiannis et al. [3] (which proves an improved upper bound on the competitiveness of the on-line coloring algorithm First-Fit in disk graphs which are graphs representing overlaps of disks on the plane) and Miyamoto et al. [14] (which constructs multi-colorings of unit disk graphs represented on triangular lattice points).

There are also several papers on coloring restricted to planar graphs of which we note Ghosh et al. [8] because it is concerned with a self-stabilizing algorithm for coloring such graphs. Their algorithm achieves a 6 coloring by transforming the planar graph into a DAG of out-degree at most five. However, this algorithm needs the full knowledge of the topology of the graph. The specificity of the problem for ad-hoc networks requires a different approach. An ad-hoc network can be a very large dynamic system, and in some cases a node can join or leave a network at any time. Thus, the full knowledge of the topology of an ad-hoc network might not be available, or possible for each node of the network at all times. Thus algorithms that can make computations in a fully distributed manner, using in each node only information about the network within a fixed distance neighborhood of the node, are of particular interest in ad hoc networks. Examples of algorithms of this type are the Gabriel test [6] for constructing a planar spanner, face routing [11],[1], or an approximation of the minimum spanning tree [22], [4].

To reduce network complexity, the unit disk graph  $G$  is sometimes reduced to a much smaller subset  $P$  of its edges called a spanner. A good spanner must have some properties so that certain parameters of communication within  $P$  are preserved. To ensure all to all communication  $P$  must be connected. An important property is having a constant stretch factor  $s$ , guaranteeing that the length of a path joining two nodes in  $G$  is at most  $s$  times shorter than the shortest path joining these nodes in  $P$ . A desired property of  $P$  is planarity, which, on one hand, permits an efficient routing scheme based on *face routing* and, on the other hand, ensures linear complexity of  $P$  with respect to its number of nodes. Planar graphs also have low chromatic number, hence a small set of frequencies is sufficient to realize radio communication.

In this paper, we are interested in *local* distributed coloring algorithms whereby messages emanating from any node can propagate for only a constant number of hops. This model was first introduced in the seminal paper of Linial [12]. One of the advantages of this model is that it aims to obtain algorithms that could cope with a dynamically changing infrastructure in a network. In this approach, each node may communicate with nodes at a bounded distance from it and thus a local change in the network only needs a local adjustment of a solution. In Linial's model of communication, locality results in a constant-time distributed algorithm.

Several research papers along these lines include Szegedy et al. [16], Vredeveld et al. [21] just to mention a few. The previously mentioned papers ignore the geometric nature of the resulting wireless ad hoc network, as this might be the case, e.g., when the nodes are equipped with GPS devices. In our approach, we assume that the nodes of the network are placed in the plane according to some global system of coordinates and that each node knows its proper coordinates, i.e. the network nodes are *geographically aware*. In the particular case of geographically aware ad hoc mobile networks, the techniques based on the traditional coloring algorithms for planar graphs (see the book of Jensen and Toft [10][Chapter 2]) like the celebrated 4-coloring theorem for planar graphs of Appel and Haken (see also Robertson et al [15]) may not be effective or not adaptable to the distributed setting. Our approach in this paper is to use geometric awareness of the underlying ad hoc network in order to achieve a fully distributed, local coloring algorithm.

## 1.1 Network model and results of the paper

We are given a set  $S$  of  $n$  points in the plane and a planar subgraph  $G$  of the unit disk graph induced by  $S$ . We assume that  $G$  is connected and all the nodes either know their exact  $(x, y)$  coordinates (which could be achieved for example by having the nodes equipped with GPS devices) or have a consistent relative coordinate system (know their  $(x, y)$  coordinates relative to each other, see Capkun et al. [2]). If  $G$  is not connected, all reasoning can be applied to each connected component of  $G$  independently, in which case the coloring constructed applies to each connected component. We propose two local coloring algorithms. The first simple algorithm computes a 9-coloring of the planar graph. We assume that each node knows its 9-neighborhood in the original unit disk graph (i.e. all nodes at distance at least 9 hops away from it), it can communicate with each node of its neighborhood and it is aware which of these nodes belong to the planar subgraph. We then present a more complicated algorithm whereby each vertex can compute its color in a 7-coloring of the planer graph using only information on the subgraph located within at most a constant number  $h = 201$  of hops away from it. The constant  $h$  is quite large though in practice nodes at much smaller distance will need to communicate. The algorithm does not determine locally either what the different connected components are or even what are the local parts of a component connected somewhere far away. Moreover, the correctness of the algorithm is independent of the connectivity of the planar subgraph.

## 2 Simple Local Coloring Algorithm

The basic idea of the coloring algorithms in this paper is to partition the plane containing  $G$  into fixed sized areas, compute a coloring of the subgraph of  $G$  within each such area independently and possibly adjust colors of some vertices that are on the border of an area and thus are adjacent to nodes in another area. This is possible to do consistently and without any pre-processing because the nodes know their coordinates and thus can determine the area in which they belong. Since each area is of fixed size, a subgraph of the given unit disk graph belonging to this area is of a bounded diameter. Hence a constant number of hops is needed for a node to communicate with each other node of the same area.

### 2.1 Coloring with regular hexagonal tilings

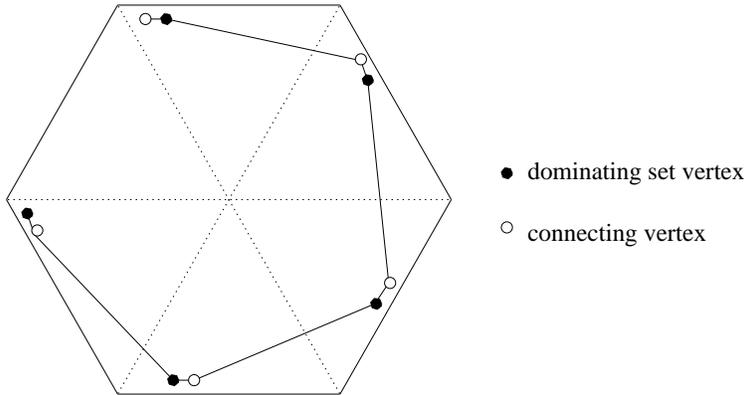
The simplest partitioning we consider is obtained by tiling the plane with regular hexagons having sides of size 1. We suppose that two edges of the hexagons are horizontal and one of the hexagons is centered in coordinates  $[0, 0]$ . To assure the disjointness of the hexagon areas we assume that only the upper part of the boundary and the leftmost vertex belongs to each hexagon area while the rightmost vertex does not belong to it. Under such conditions, two vertices of  $G$  can be connected by an edge only if they are in the same or adjacent hexagon areas.

As each vertex knows its own coordinates, it can calculate which hexagon it belongs to. In the first step each vertex communicates with vertices within its hexagon and learns the part of the subgraph located in its hexagon. By Lemma 1, communication inside a hexagon may be done using at most nine hops.

**Lemma 1.** *Any connected component of the subgraph of a unit disk graph induced by its nodes belonging to a regular hexagon with sides of size 1 has a diameter smaller or equal to nine. Moreover, there exist configurations of nodes inside the hexagon of diameter equal to nine.*

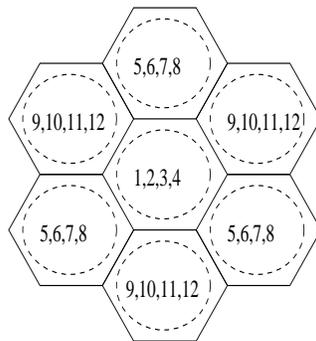
*Proof.* We prove first that any independent set of nodes centered inside a given hexagon consists of at most five nodes. Indeed, every node of such an independent set covers an angular sector, originated at the center of the hexagon, of at least  $120^\circ$  and two independent nodes may share an angular sector of at most  $60^\circ$ . Consequently, at most five elements of an independent set may be centered in the external region.

To prove the claim of the lemma take two nodes of a hexagon furthest apart and the shortest path joining them. Odd-numbered nodes of such path form an independent set. Since any such set contains at most five nodes, at most nine edges may belong to any shortest path inside the hexagon. Figure 1 shows an example of a configuration of nodes proving that nine is also a lower bound for the diameter problem.



**Fig. 1.** Vertices within a hexagon of radius 1 for which the unit disk graph is of diameter nine.

We could apply the standard 4-coloring algorithm for each connected component of the graph induced by  $G$  on points within each hexagon. Since the tiling of the plane by hexagons can be 3-colored, three disjoint sets of four colors can be used, one set of colors for vertices in each hexagon area of the same color. This would lead to a 12 coloring of  $G$  (see the coloring scheme of the hexagonal



**Fig. 2.** Coloring of a hexagonal tiling of the plane using 12 colors.

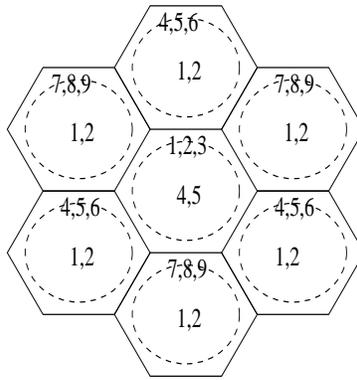
tiling depicted in Figure 2).

The number of colors can be reduced by coloring the outer face of each component in a hexagon with prescribed colors, using the result of the following theorem.

**Theorem 1 (3 + 2 Coloring, Thomassen [17]).** *Given a planar graph  $G$ , 3 prescribed colors and an outer face  $F$  of  $G$ , graph  $G$  can be 5-colored while the vertices of  $F$  use only the prescribed three colors.*

The precise result in Thomassen's paper [17] (see also Dörre [5]) states that every planar graph is  $L$ -list colorable for every list assignment with lists of size 5. The proof of this result however implies the following statement taken from Tuza et al. [19] that for a planar graph  $G$  with outer face  $F$ , every pre-coloring of two adjacent vertices  $v_1, v_2$  of  $F$  can be extended to a list coloring of  $G$  for every list assignment with  $|L(v)| = 3$  if  $v \in V(F) - \{v_1, v_2\}$  and  $|L(v)| = 5$  if  $v \notin V(F)$ , where  $V(F)$  denotes the vertex set of  $F$ . This is what we are using, although we do not pre-color any two vertices (See also the related paper of Voigt [20] which states that list coloring of planar graphs is impossible for some graphs if lists are of length at most 4.)

Using Theorem 1 the number of colors can be reduced to 9 as follows. The idea is that the vertices of  $G$  in a hexagon can be adjacent only to the outer face vertices of the graphs induced in the neighboring hexagons. Three disjoint sets of colors of size three are used as the prescribed colors of the vertices on the outer faces, the inner vertices of  $G$  in a hexagon can employ, in addition to the three colors used on the outer face of its hexagon, two additional colors of the outer faces of other hexagons (see Figure 3). Thus we obtain the following



**Fig. 3.** Colorings of a hexagonal tiling of the plane using 9-colors.

result.

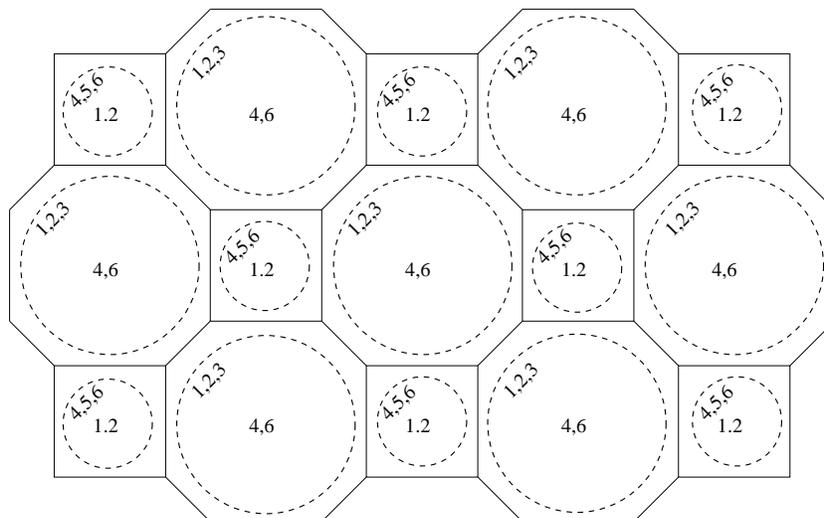
**Theorem 2.** *Using the partitioning of a plane into regular hexagons with sides of size 1, a vertex can compute its color in a 9-coloring of the planar graph using only information on the subgraph located within at most 9 hops away from it.*

### 3 The 7-coloring Algorithm

In this section we give a 7 coloring algorithm and prove its correctness. The trade-off is the larger area of the network a node needs to examine in this algorithm and a more complex partitioning of the plane. To begin with, we attempt to reduce the number of colors by using a tiling consisting of two tile types. The subgraph in each tile is colored using 5 colors as in Theorem 1, but we shall use six colors in all the tiles together. We then show how to adjust locally the coloring on the borders of tiles where some improper coloring could exist. This can be done using one additional color for a total of seven colors.

#### 3.1 Reducing the number of colors using mixed tilings

We will employ the tiling using octagons and squares shown in Figure 4. Each square is of size  $5 + \epsilon$  while the slanted part of an octagon border is of length  $3 + \epsilon$ , meaning that these sides can be chosen arbitrarily close to but greater than 5 and 3, respectively. We shall assume that one of the octagons is centered



**Fig. 4.** A coloring of the set of points based on the octagon/square tiling of the plane.

in coordinates  $[0,0]$ . The reasons for choosing the sizes of tiles this way is to isolate the meeting places of octagons (the slanted border part) from each other so that each of them can be dealt with independently and locally, and to ensure that there is no edge between two vertices in different squares or between two vertices of the same squares that could be recolored for different crossings in the algorithm. In handling a meeting place, only vertices at a distance at most

2 from it will be recolored, therefore it is impossible to have two neighboring vertices recolored due to different crossing.

Similarly as in the previous coloring, the subgraph induced by the vertices in a tile is colored using 5 colors, three of them are used on the outer face and these three colors plus the additional two colors are used on vertices not on the outer face using Theorem 1. Since each node knows its location, it can determine which tile it belongs to. By Lemma 2 the communication inside an octagon may be done using at most 201 hops.

**Lemma 2.** *Any subgraph of a unit disk graph induced by its vertices belonging to an octagon used in the algorithm has a diameter smaller or equal to 201.*

*Proof.* The largest independent set of unit disks (i.e. disks of radius one) centered inside the octagon equals the largest possible set of disjoint disks of diameter one that may be centered within the octagon. To bound the size of such an independent set it is sufficient to find the best packing of such disks in the enlarged octagon, which is the result of the Minkowski sum of the octagon and the disk of diameter one. The surface of the octagon is slightly greater than  $(5 + 3\sqrt{2})^2 - 2(3\sqrt{2}/2)^2$  and the surface of its enlargement belt equals  $16 + \pi/4$ , which results in the total surface of around 79.6. By comparing it to the surface of the disk, equal to  $\pi/4 = 0.785$ , we see that no more than 101 such disks of diameter one may be packed within the enlarged octagon. Hence, any independent set of unit disks centered within the octagon contains at most 101 elements. We can complete such an independent set to a shortest path of length at most 201.

Despite the fact that the simple, surface comparing argument leaves some room for improvement (the packing density is at most  $\pi\sqrt{3}/6 = 0.907$ ), it is possible to construct configurations of nodes, centered inside the octagon, inducing a graph of diameter at least 183.

Since the square tile admits a smaller hop diameter, any node can determine the subgraph induced by the vertices in its tile by examining nodes at hop distance at most 201.

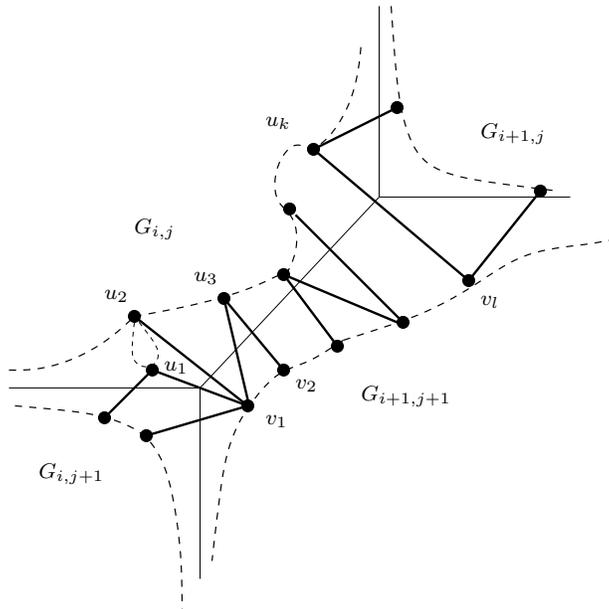
The color sets used in the tiles are as specified in Figure 4. The resulting coloring is using only 6 colors. Due to the chosen sizes of the tiles and the chosen coloring scheme, an edge of  $G$  crossing from a square to an octagon is between vertices of different colors. After this initial color assignment, any edge of  $G$  whose endpoints are of the same color is necessarily an edge crossing the slanted part of the border of two adjacent octagons. The following construction shows that using one additional color and with careful attention to detail near the common border of adjacent octagons, some of the vertices can be recolored in order to achieve a 7-coloring of  $G$ . Details follow in Subsection 3.2.

### 3.2 Adjusting the coloring

As seen from Figure 4, the centers of the tiles form an infinite regular mesh. We shall denote the hexagon tile that is centered at coordinates  $[0, 0]$  as  $S_{0,0}$ . For a tile denoted  $S_{i,j}$ , its horizontal left, horizontal right, vertical down, vertical

up neighboring tile is denoted  $S_{i-1,j}$ ,  $S_{i+1,j}$ ,  $S_{i,j+1}$ ,  $S_{i,j-1}$ , respectively. Let  $G_{i,j}$  denote the subgraph of  $G$  induced by the vertices located within  $S_{i,j}$  and suppose that  $F_{i,j}$  denotes the outer face of  $G_{i,j}$ . In case that  $G_{i,j}$  is not connected, we consider each connected component of  $G_{i,j}$  separately. (Notice that we only need to consider those components of  $G_{i,j}$  that contain vertices adjacent to more than one octagon, for otherwise the coloring of such a component could be added to the coloring of  $G_{i+1,j+1}$ .)

Let  $S_{i,j}$  be one of the hexagonal tiles, i.e.,  $i + j$  is even. Consider the place where  $S_{i,j}$  and  $S_{i+1,j+1}$  meet (we call it a  $CR_{i,j}$  crossing). Consider the sequence of vertices obtained in a counterclockwise cyclical traversal of the outer face  $F_{i,j}$  of  $G_{i,j}$  and let  $L = \{u_1, u_2, \dots, u_k\}$  be the *shortest subsegment* of this traversal containing all the vertices connected to  $G_{i+1,j+1}$ , i.e., as in Figure 5.



**Fig. 5.** A typical simple crossing.

Notice that  $L$  could contain some vertex more than once if the outer face is not a simple curve. Define  $M = \{v_1, v_2, \dots, v_l\}$  analogously for  $G_{i+1,j+1}$ , using clockwise traversal.

We say that crossing  $CR_{i,j}$  is *simple* if no inside vertex of  $L$  different from  $u_1$  and  $u_k$  is connected both to a vertex of  $M$  and to a vertex in  $G_{i,j+1}$  or  $G_{i+1,j}$ , and the same analogous condition holds for the inside vertices of  $M$ . If the crossing is simple, the problem of having some edges between vertices of  $L$  or  $M$  having both endpoints of the same color can be resolved using the following lemma.

**Lemma 3.** *Let  $CR_{i,j}$  be a simple crossing (see Figure 5). Then the vertices of  $L$  and  $M$  and some of the neighbors of  $u_1, v_1, u_k$  and  $v_k$  can be recolored, possibly with the help of color 7, in such a way that no edge incident to  $L$  or  $M$  or a neighbor of either of  $L$  or  $M$  has both endpoints of the same color.*

*Proof.* After the initial coloring, the only edges which might have endpoints of the same color are the edges connecting vertices of  $L$  to vertices of  $M$ . Without loss of generality we may assume that the vertices of  $L$  and  $M$  use colors 1, 2, 3, the inside of  $G_{i,j}$  uses in addition colors 4, 5, while the inside of  $G_{i+1,j+1}$  uses colors 4, 6 in addition to 1, 2, 3.

The recoloring is done in two steps, where in the first step the conflict vertices of  $L$  and  $M$  (i.e. the ones with a neighbor of the same color) are recolored as indicated in Table 1.

Conflict vertex of	old color	new color
$L$	1	6
$M$	2	5
$L$	3	7

**Table 1.** First step of recoloring

This ensures that no edge incident to an *inner vertex* of  $L$  or  $M$ , i.e. different from  $\{u_1, v_1, u_k, v_l\}$ , has both endpoints of the same color since:

- if there was an edge from  $L$  to  $M$  connecting two vertices of the same color, one endpoint of this edge has been recolored.
- As no color 6 was used in  $G_{i,j}$ , the vertices of  $L$  recolored to 6 have no neighbors of color 6 in  $G_{i,j}$  (and they cannot be neighbors, as both had color 1 in the coloring of  $G_{i,j}$ ). They also do not have neighbors of color 6 in  $G_{i+1,j+1}$  as all their neighbors in  $G_{i+1,j+1}$  are on the outer face and thus of colors 1, 2, 3 (and newly 5). Finally, from the second property of simple crossing it follows that the inner conflict vertices of  $L$  and  $M$  do not have neighbors in  $G_{i,j+1}$  and  $G_{i+1,j}$ .
- Analogous argument applies for vertices recolored to 5 in  $M$  and vertices recolored to 7 in  $L$ .

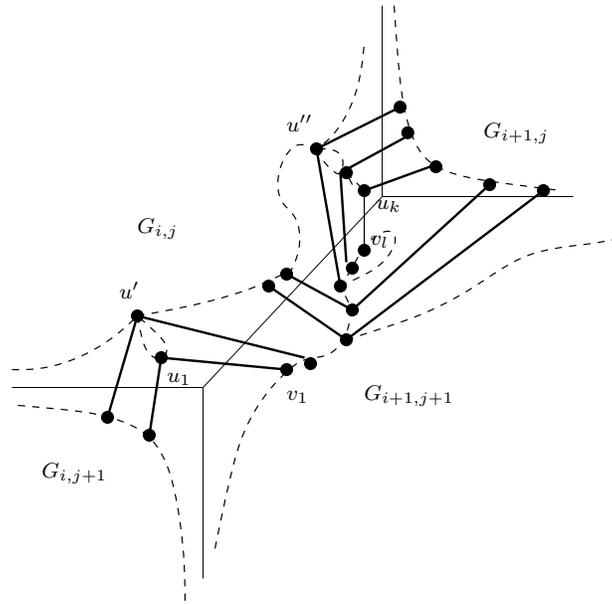
It remains to consider edges incident to  $\{u_1, v_1, u_k, v_l\}$  that might have endpoints of the same color; for example  $u_1$  was recolored from 1 to 6 but it has a neighbor of color 6 in  $S_{i,j+1}$  (note that there is no problem if the new color was 7). In such a case, these same-color neighbors in  $S_{i,j+1}$  are recolored to color 7. We claim that this does not create same-color edges. First note that if  $u_1$  recolored its neighbors of color 6 in  $G_{i,j+1}$ , then  $v_1$  necessarily kept its original color, since  $v_1$  might change its color only if it was originally 2. Hence, by the simplicity of the crossing, only the neighbors of  $v_1$  of color 6 (or, by symmetry, only the neighbors of  $u_1$  of color 5) need to be recolored to color 7. The cases

for other extreme vertices of  $L$  and  $M$  are analogous. Since the width of the gap between the squares is greater than 3, it ensures that the recoloring of vertices in  $G_{i,j+1}$  and  $G_{i+1,j}$  does not create any conflict in coloring.

Furthermore, any two vertices of  $G_{i,j+1}$  which were recolored to 7 due to different crossings with octagons cannot be neighbors since the size of the squares is  $5 + \epsilon$ .

While the width of the gap between the squares ensures the first condition for a crossing to be simple is always satisfied, there can be a case when several different vertices of  $L$  are connected both to a vertex of  $M$  and to a vertex in  $G_{i,j+1}$  or  $G_{i+1,j}$ , see Figure 6. Notice that this may happen when some inner vertices of  $L$  or  $M$  are cut vertices in  $G_{i,j}$  or  $G_{i+1,j+1}$ .

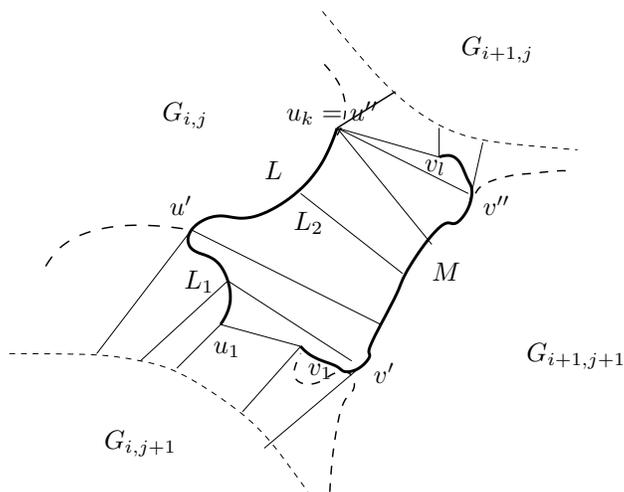
We resolve the problem of a crossing that is not simple by a pre-processing phase in which some of the vertices in the octagons are assigned to the neighboring squares, with the goal to make the crossing simple.



**Fig. 6.** A not simple crossing.

Consider a crossing  $CR_{i,j}$  and let  $L$  and  $M$  be defined as before. Let  $u'$  be the last (in  $L$ ) occurrence of a node connected to both  $M$  and  $G_{i,j+1}$  (if there is no such node, set  $u' = u_1$ ). Similarly, let  $u''$  be the first occurrence in  $L$  of a node connected to both  $M$  and  $G_{i+1,j}$ . Define  $v'$  and  $v''$  analogously in  $M$ , using clockwise traversal, see Figure 6. Any vertex of  $L$  which is connected to both  $G_{i,j+1}$  and  $G_{i+1,j+1}$  must occur in the segment of  $L$  from  $u_1$  to  $u'$ , since

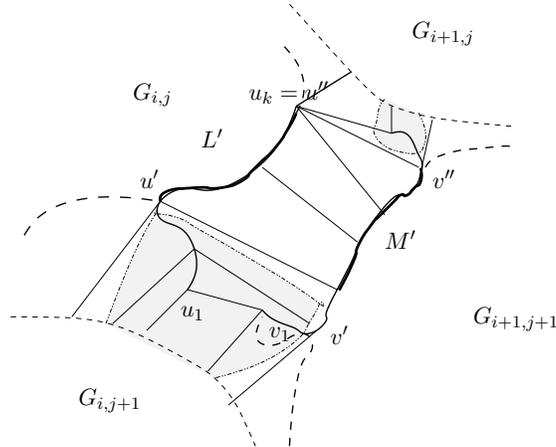
the edges incident with  $u'$  connecting it to  $G_{i,j+1}$  and  $G_{i+1,j+1}$  act as separators in the planar graph. Similarly, any node of  $L$  which is connected to both  $G_{i+1,j}$  and  $G_{i+1,j+1}$  must occur in the segment of  $L$  from  $u''$  to  $u_k$  (see Figure 7).



**Fig. 7.**  $L$  and  $M$  in a crossing.

We now partition  $L$  into three parts: Let  $L_1$  be the shortest initial segment of  $L$  from  $u_1$  to the first occurrence of  $u'$  so that all vertices connected to both  $G_{i,j+1}$  and  $G_{i+1,j+1}$  are contained in  $L_1$ , let  $L_3$  be the shortest final segment of  $L$  starting with an occurrence of  $u''$  so that all vertices connected to both  $G_{i+1,j}$  and  $G_{i+1,j+1}$  are contained in  $L_3$ , and  $L_2$  be the remaining part of  $L$ . We define  $M_1$ ,  $M_2$ , and  $M_3$  analogously as segments of  $M$  using  $v'$  and  $v''$ .

To make the crossing simple, we assign the components of  $G_{i,j}$  separated by  $u'$  and encountered in the traversal of  $L_1$  to  $G_{i,j+1}$ , and the components of  $G_{i,j}$  separated by  $u''$  and encountered in the traversal of  $L_3$  are assigned to be  $G_{i+1,j}$ . The same is applied to the segments of  $G_{i+1,j+1}$  separated by  $v'$  and  $v''$  and encountered in the traversal of  $M_1$  and  $M_3$ , (see Figure 8). All the components that are assigned to  $G_{i,j+1}$  are inside the area bordered by the edges connecting  $u'$  or  $v'$  to  $G_{i+1,j+1}$  and  $G_{i,j+1}$  or  $G_{i,j}$  and  $G_{i,j+1}$ . Similarly all the components that are assigned to  $G_{i+1,j}$  are inside the area bordered by the edges connecting  $u''$  or  $v''$  to  $G_{i+1,j+1}$  and  $G_{i+1,j}$  or  $G_{i,j}$  and  $G_{i+1,j}$ . Since the length of the crossing is more than 3, there cannot be any edge between vertices assigned to  $G_{i+1,j}$  and  $G_{i,j+1}$ . Furthermore, after this reassignment,  $u'$  is the only vertex in  $G_{i,j}$  that can be connected to both  $G_{i+1,j+1}$  and  $G_{i,j+1}$  and  $u''$  is the only vertex in  $G_{i,j}$  that can be connected to both  $G_{i+1,j+1}$  and  $G_{i+1,j}$ . The analogous statement can be made about  $v'$  and  $v''$ . Thus the crossing  $L'$  and  $M'$  between  $G_{i,j}$  and  $G_{i+1,j+1}$  is a subset of  $u', L_2, u''$  and  $v', M_2, v''$  and this



**Fig. 8.**  $L'$  and  $M'$  after the transformation. The shaded areas belong to  $G_{i,j+1}$  and  $G_{i+1,j}$ .

modified crossing  $CR_{i,j}$  satisfies both conditions of a simple crossing, see Figure 8, and we can thus proceed with the coloring as stated in Lemma 3.

The following lemma, together with the size of the squares being selected as  $5 + \epsilon$ , allows us to apply Lemma 3 to each crossing independently.

**Lemma 4.** *Any vertex recolored due to resolving conflicts in crossing  $CR_{i,j}$  is at a distance at most 2 from the line separating  $S_{i,j}$  and  $S_{i+1,j+1}$ .*

*Proof.* Since  $L_2$  and  $M_2$  are subsets of the original  $L$  and  $M$ , they are at distance at most 1 from the separating line. From construction in Lemma 3, only the vertices of  $L_1$  and  $M_2$  and some of their neighbors are recolored.

Notice that the hexagon is of diameter less than 10.6 and a square, even after the inclusion of the relocated parts is of diameter less than 9.1. Hence, by Lemma 2, the local subgraphs which need to exchange the information needed to compute the coloring are of diameter at most 201.

### 3.3 Local 7-coloring algorithm

Putting the pieces together we have the following local, fully distributed algorithm that is executed at each vertex of the graph to obtain a valid 7-coloring of the graph.

The results of this section can be summarized in the following theorem.

**Theorem 3.** *Given a planar subgraph of the unit disk graph whose vertices correspond to hosts that are each aware of its geometric location in the plane, Algorithm 1 computes locally a 7-coloring of this subgraph using only information on the subgraph located within a constant number of hops away from it.*

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**Algorithm 1** The local 7-coloring algorithm for a vertex  $v$

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- 1: Learn your neighborhood up to distance 201 // *Note that all steps can be performed locally using the information learned in the first (communication) step, without incurring further communication.*
  - 2: From your coordinates, identify the square/octagon  $S_{i,j}$  you are located in, and calculate the connected component of  $G_{i,j}$  you belong to.  
// *The next step is for vertices near a crossing*
  - 3: Calculate  $L$  and  $M$ , and then  $L'$  and  $M'$ . Determine whether you have been shifted to a neighboring square. Determine whether  $L'$  and  $M'$  are connected, if not but the squares are now connected, repeat the process until the final  $L^*$  and  $M^*$  are computed.
  - 4: Apply the 3 + 2 coloring algorithm from Theorem 1 for each  $G_{i,j}$ , as in Figure 4
  - 5: Apply the recoloring from Lemma 3.
- 

## 4 Conclusions

We gave two algorithms computing locally a coloring of a given planar subgraph of a unit disk graph. The two algorithms differ in the number of colors used and in the size of the neighborhood they need to know. Two interesting questions remain for further studies:

1. whether or not one can substantially decrease the size of the subgraph used in the calculation of a 7-coloring, and
2. whether one can decrease the number of colors that are sufficient for coloring a given planar subgraph of a unit disk graph while maintaining the algorithm's locality and full distributivity.

In particular, is it possible to 4-color a planar graph using a local algorithm? Since there exist uniquely 4-colorable planar graphs it seems unsuitable to start the 4-coloring independently in the different parts of the graph. However, the fundamental nature of any local algorithm is to perform the local computations independently. The interesting open question is then the lower bound on the number of colors needed by a local algorithm to perform a coloring of a planar subgraph of a unit disk graph.

## References

1. P. Bose, P. Morin, I. Stojmenovic, and J. Urrutia. Routing with guaranteed delivery in ad hoc wireless networks. *Wireless Networks*, 7:609–616, 2001.
2. S. Capkun, M. Hamdi, and J.-P. Hubeaux. GPS-free positioning in mobile ad hoc networks. *Cluster Computing*, 5:157–167, 2002.
3. I. Caragiannis, A. V. Fishkin, C. Kaklamanis, and E. Papaioannou. A tight bound for online coloring of disk graphs. In A. Pelc and M. Raynal, editors, *SIROCCO 2005, Mont Saint-Michel, France, May 24-26, 2005, Proceedings*, volume 3499 of *Lecture Notes in Computer Science*, pages 78–88. Springer, 2005.

4. E. Chavez, S. Dobrev, E. Kranakis, J. Opatrny, L. Stacho, and J. Urrutia. Local construction of planar spanners in unit disk graphs with irregular transmission ranges. In M. K. j. Correa, A. Hevia, editor, *Latin 06*, volume 3387 of *Lecture Notes in Computer Science*, pages 286–297. Springer Verlag, 2006.
5. P. Dörre. Every planar graph is 4-colourable and 5-choosable a joint proof. Fachhochschule Südwestfalen (University of Applied Sciences), unpublished note.
6. K. R. Gabriel and R. R. Sokal. A new statistical approach to geographic variation analysis. *Systemic Zoology*, 18:259–278, 1972.
7. P. Galinier and A. Hertz. A survey of local search methods for graph coloring. *Computers & OR*, 33:2547–2562, 2006.
8. S. Ghosh and M. Karaata. A self-stabilizing algorithm for coloring planar graphs. *Distributed Computing*, 7:55–59, 1993.
9. A. Gräf, M. Stumpf, and G. Weißenfels. On coloring unit disk graphs. *Algorithmica*, 20(3):277–293, 1998.
10. T. R. Jensen and B. Toft. *Graph coloring problems*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, 1995.
11. E. Kranakis, H. Singh, and J. Urrutia. Compass routing on geometric networks. In *Proc. of 11th Canadian Conference on Computational Geometry*, pages 51–54, August 1999.
12. N. Linial. Locality in distributed graph algorithms. *SIAM J. COMP.*, 21(1):193–201, 1992.
13. M. V. Marathe, H. Breu, H. B. Hunt III, S. S. Ravi, and D. J. Rosenkrantz. Simple heuristics for unit disk graphs. *Networks*, 25(1):59–68, 1995.
14. Y. Miyamoto and T. Matsui. Multicoloring unit disk graphs on triangular lattice points. In *SODA*, pages 895–896. SIAM, 2005.
15. N. Robertson, D. P. Sanders, P. Seymour, and R. Thomas. Efficiently four-coloring planar graphs. In *Proceedings of the Twenty-Eighth Annual ACM Symposium on the Theory of Computing*, pages 571–575, Philadelphia, Pennsylvania, May 1996.
16. M. Szegedy and S. Vishwanathan. Locality based graph coloring. In *Proceedings of the 25th Annual ACM Symposium on the Theory of Computing*, pages 201–207, May 1993.
17. C. Thomassen. Every planar graph is 5-choosable. *Combinatorial Theory Series B*, 62(1):180–181, Sep. 1994.
18. Z. Tuza. Graph colorings with local constraints: A survey. *Discussiones Mathematicae–Graph Theory*, 17(2):161–228, 1997.
19. Z. Tuza and M. Voigt. A note on planar 5-list colouring: non-extendability at distance 4. *Discrete Mathematics*, 251(1):169–172, 2002.
20. M. Voigt. List colourings of planar graphs. *Discrete Mathematics*, 120(1-3):215–219, 1993.
21. T. Vredeveld and J. K. Lenstra. On local search for the generalized graph coloring problem. *Oper. Res. Lett*, 31(1):28–34, 2003.
22. Y. Wang and X.-Y. Li. Localized construction of bounded degree and planar spanner for wireless ad hoc networks. In *DialM: Proceedings of the Discrete Algorithms and Methods for Mobile Computing & Communications*, 2003.