COMP 2804 — Assignment 2

Due: Wednesday February 24, before 4:30pm, in the course drop box in Herzberg 3115. Note that 3115 is open from 8:30am until 4:30pm.

Assignment Policy: Late assignments will not be accepted. Students are encouraged to collaborate on assignments, but at the level of discussion only. When writing the solutions, they should do so in their own words. Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams. Please show your work for each question.

Important note: When writing your solutions, you must follow the guidelines below.

- The answers should be concise, clear and neat.
- When presenting proofs, every step should be justified.
- Assignments should be stapled or placed in an unsealed envelope.
- All questions carry equal marks.

Substantial departures from the above guidelines will not be graded.

Question 1: On the first page of your assignment, write your name, student number, and course number COMP 2804.

Question 2: A small computer network consists of 6 computers. Each computer is directly connected to 0 or more computers. Show that there are at least two computers in the network that are directly connected to the same number of other computers. Is the statement true if we have 10 computers instead of 6?

Question 3: Show that $a_n = n^2 + n + 1$ satisfies

$$\begin{cases} a_0 = 1 \\ a_k = a_{k-1} + 2k \quad \text{for } k > 0. \end{cases} \tag{1}$$

Question 4: Consider the following recursive function defined for positive values of n

$$M(n) = \begin{cases} n - 10 & \text{if } n > 100 \\ M(M(n + 11)) & \text{if } n \leq 100 \end{cases} \tag{2}$$

Evaluate $M(120)$, $M(111)$, $M(97)$, $M(94)$. Do you observe something interesting for the value of $M(n)$ for $n \leq 100$?

Question 5: Let d_n denote the number of ways that n-letters can be put into n-envelopes so that no letter goes into the correct envelope. Show that $d_1 = 0$, $d_2 = 1$, $d_3 = 2$, and in general for $n \geq 3$,

$$d_n = (n-1)(d_{n-1} + d_{n-2}).$$

Question 6: Using induction for $n \geq 1$, show that d_n in the previous question can be expressed as

$$d_n = n!\Big(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots + (-1)^n \frac{1}{n!}\Big).$$

Question 7: Exercise 4.36 from the Course Text Book by Michiel Smid.

Question 8: Give a recursive definition of the set of integers that are multiples of 3. Show that your definition indeed generates all the elements of the set $\{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}$.

Question 9: Show that

$$a_n = \frac{1 - r^{n+1}}{1 - r}, r \neq 1$$

satisfies the recurrence relation

$$\begin{cases} a_0 = 1 \\ a_k = a_{k-1} + r^k \quad \text{for } k > 0. \end{cases} \tag{3}$$

Question 10: A binary tree is
- either one single node
- or a node whose left subtree is a binary tee and whose right subtree is a binary tree.
Show that any binary tree with n leaves has exactly $2n - 1$ nodes.

Question 11: (Bonus Problem:) Assume you have a set $A = \{a_1, a_2, \ldots, a_{n+1}\}$ of $n + 1$ positive numbers such that $\sum_{i=1}^{n+1} a_i = 2n$. Prove or disprove that for any integer k, where $1 \leq k \leq 2n$, we can always find a subset $B \subseteq A$ such that the sum of elements of B equals k.