Lecture 1

INTRODUCTION

LOGISTICS

MOTIVATING PROBLEMS

- RAMSEY THEORY
- SPERNERS THEOREM
- QUICKSORT

COMP 2804

DISCRETE STRUCTURES

II

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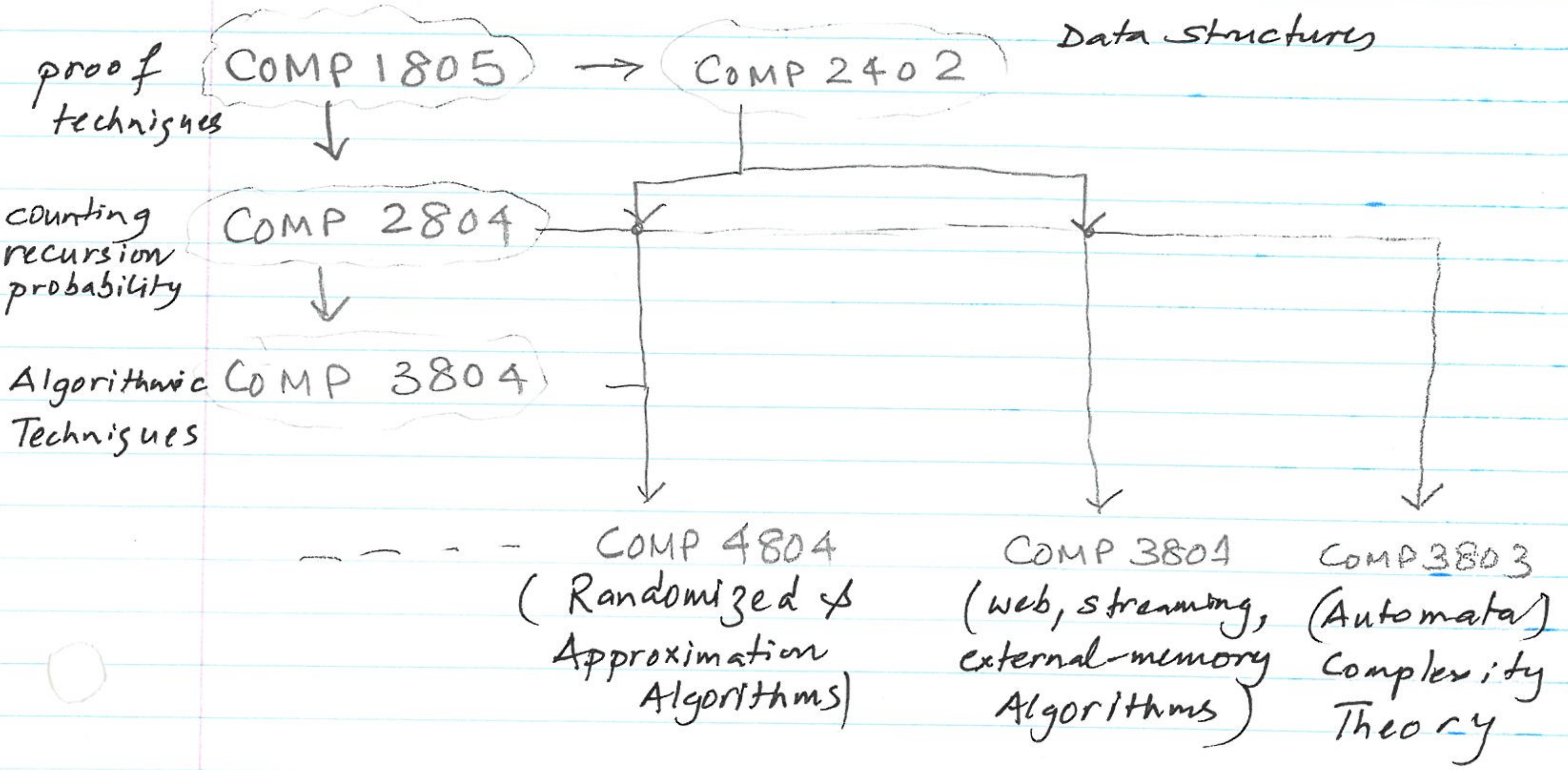

Discrete Structures II

- Branch of Mathematics dealing with countable sets.
- Integers, Graphs, Rational Numbers.
- Computer programs & algorithms manipulate discrete structures

e.g. searching, sorting, street maps,

query on the web, recommendation systems, - - -

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Text-book

Discrete Structures for Computer Science: Counting, Recursion, and Probability

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- HISTORY of the course

- Evaluation

4 Assignments	25%
MID-TERM	25%
FINAL	50%

- Counting 2-3 weeks

  Recursion 2-3 weeks

  Probability : 4-6 weeks

- Course on Problem Solving

  ⇒ Solve Problems

  ⇒ Learn how to solve problems

  ⇒ Important to hit road-blocks and learn how to overcome them.

- CHAPTER 2 is Home Work

BACKGROUND MATERIAL

Chapter 2

Mathematical Preliminaries

2.1 Basic Concepts

Throughout this book, we will assume that you know the following mathematical concepts:

1. A *set* is a collection of well-defined objects. Examples are (i) the set of all Dutch Olympic Gold Medallists, (ii) the set of all pubs in Ottawa, and (iii) the set of all even natural numbers.

2. The set of *natural numbers* is $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

3. The set of *integers* is $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

4. The set of *rational numbers* is $\mathbb{Q} = \{m/n : m \in \mathbb{Z}, n \in \mathbb{Z}, n \neq 0\}$.

5. The set of *real numbers* is denoted by $\mathbb{R}$.

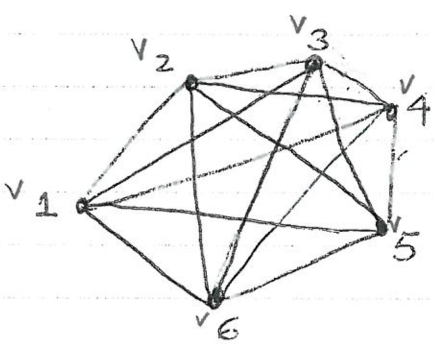
6. The *empty set* is the set that does not contain any element. This set is denoted by $\emptyset$.

7. If A and B are sets, then A is a *subset* of B, written as $A \subseteq B$, if every element of A is also an element of B. For example, the set of even natural numbers is a subset of the set of all natural numbers. Every set A is a subset of itself, i.e., $A \subseteq A$. The empty set is a subset of every set A, i.e., $\emptyset \subseteq A$. We say that A is a *proper subset* of B, written as $A \subset B$, if $A \subseteq B$ and $A \neq B$.

L1 - COMP 2804

1. RAMSEY THEORY

- Consider a complete graph on 6-vertices, where each edge is colored red or blue. Then there is a red triangle or a blue triangle



$|V| = 6$

$|E| = 15$

$R[3,3] = 6!$

$R[3,3,3] = 17$

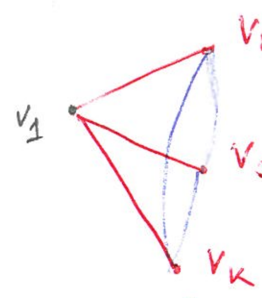
<u>Proof</u>: Consider edges incident to vertex v_1.

There are 5 edges $\{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_1, v_5)\}$.

Since each edge is colored red or blue, there are at least three edges of the same color.

WLOG, let them be red.

Let $(v_1, v_i), (v_1, v_j), (v_1, v_k)$ be red, where $i \neq j$, $i \neq k$, $k \neq j$

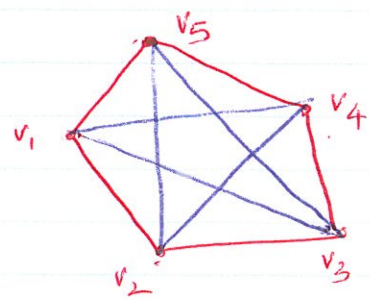


- If any of (v_i, v_j) or (v_j, v_k) or (v_i, v_k) is red then we have a red-triangle.

$\Rightarrow (v_i, v_j), (v_i, v_k), (v_j, v_k)$ are blue but then $\{v_i, v_j, v_k\}$ forms a blue triangle. $\square$

Question 1: Will this hold for complete graphs on 5-vertices?

Answer: No



Q2: Where does the proof breaks down?

Q3: What about 7 or more vertices?

Answer: Yes.

Q4: Let us say we are interested in higher order graphs, i.e in place of a red or blue triangle, we are interested in red or blue k-clique.

A <u>k-clique</u> is a complete graph on k-vertices.

Theorem 2: Let $k \geq 3$, $n \geq 3$, and $n \leq \lfloor 2^{k/2} \rfloor$.

There exist a coloring of edges of a complete graph on n-vertices by red or blue color such that it neither contains a red clique nor a blue clique.

Proof: Counting & Probabilistic Techniques.

(later in the course)

Consequence: Let $n = 2^{10}$, $k = 20$

then any group of 20 people among a set of 1024 random people, with high probability, will contain two strangers and two friends!

Q4: What if we want to color the graph by three colors: red, blue, green.

Theorem 3: Any coloring by three colors of a complete graph on 17 vertices produces a monochromatic triangle.

2. Sperner's Theorem

Let S be a set of $n \geq 1$ elements: Let $S_1, S_2, S_3, \ldots, S_m$ be m-subsets of S such that $\forall i, j: i \neq j: S_i \not\subseteq S_j$ and $S_j \not\subseteq S_i$.

Then $m \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$

e.g. Let $S = \{a, b, c, d, e\}$.

$S_1 = \{a, b\}$	$S_5 = \{b, c\}$	$S_8 = \{c, d\}$	$S_{10} = \{d, e\}$
$S_2 = \{a, c\}$	$S_6 = \{b, d\}$	$S_9 = \{c, e\}$	
$S_3 = \{a, d\}$	$S_7 = \{b, e\}$		
$S_4 = \{a, e\}$	$\cancel{S}$.		

$$m = 10 \leq \binom{5}{\lfloor \frac{5}{2} \rfloor} = \binom{5}{2} = 10$$

Note that $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ is a binomial coefficient.

$\equiv$ Number of subsets of size $\lfloor \frac{n}{2} \rfloor$ of n.

Note that any two distinct subsets S_i and S_j of same size satisfy $S_i \not\subseteq S_j$ and $S_j \not\subseteq S_i$.

Proof: Conunting + Probability

3. QUICK SORT – RUNNING TIME ?

Input: A array A of $n \geq 1$ distinct numbers

Output: Arrange elements of A in ascending order

```
Quicksort (A, Low, High)
  if Low < High then
      p := PARTITION(A, Low, High);
      QUICKSORT (A, Low, p-1);
      QUICKSORT (A, p+1, High);

PARTITION (A, LOW, HIGH)
  PIVOT := A[HIGH];
  i := LOW;
  For j := LOW to HIGH-1 do
      if A[j] < PIVOT then
          SWAP[A[i], A[j]];
          i := i+1
  SWAP (A[i], A[HIGH])
  RETURN i
```

How many steps does QUICKSORT(A, 1, n) takes ?

How to analyze this?

~~RUN~~

Idea :

Consider

Low ↙ ↙ High

A = 4 9 2 6 11 5 7

Choose 7 as the pivot

$\underbrace{4 \quad 2 \quad 6 \quad 5}_{\text{RECURSE}} \quad \boxed{7} \quad \underbrace{9 \quad 11}_{\text{RECURSE}}$

For left : choose 5 as pivot

$\underbrace{4 \quad 2}_{\text{Recurse}} \quad \boxed{5} \quad \underbrace{6}_{\text{Recurse}}$

For right : choose 11 as pivot

$\underbrace{9}_{\text{Recurse}} \quad \boxed{11}$

Choose 2 as pivot

$\boxed{2} \quad 4$

Overall: 2 4 5 6 7 9 11

Running Time: ?

It depends on the pivots.

Whether the pivot splits the problem nicely or not!!

Scenario 1: ALWAYS UNLUCKY

Suppose pivot is the largest element always!

$\#$ steps $= n-1$.

$\#$ steps $= n-2$

⋮

$\#$ Steps $= 1$

$$\text{Total} := n-1+n-2+ \cdots +1$$

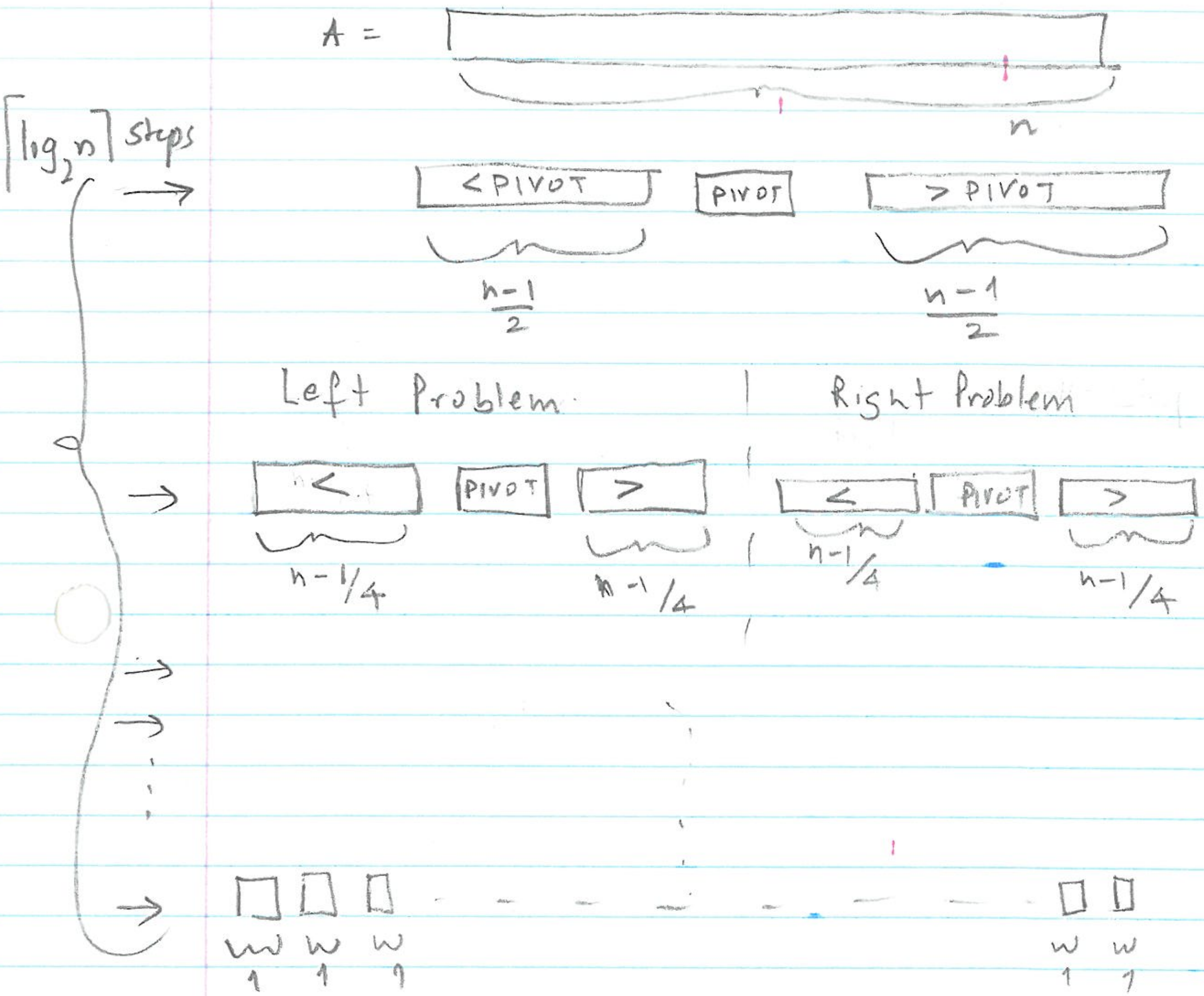
$$= \frac{n(n-1)}{2} \equiv \Theta(n^2)!$$

$\Rightarrow$ Worst-Case running time of Quicksort is quadratic in the size of the problem (n).

!

Scenario 2: ALWAYS LUCKY

Suppose pivot is always the median element



Steps ≡ n−1

Steps ≡ n−1

Steps ≡ n−1

$$\text{Total \# Steps} \equiv (n-1) * \lceil \log_2 n \rceil$$

$$\equiv O(n \log n).$$

What we will show:

If the pivot element is chosen randomly,
then the expected running time of
Quicksort is $O(n\log n)$.

Among all possible choices, choose an element,
each element has equal chance to be chosen.

Nutshell

- Math
- Discrete - counting, combinatorics, graph theory.
- Lots of probability.
- Solving Problems.