Probability

- EXPERIMENTS
- SAMPLE SPACE
- EVENTS
- RELATIVE FREQUENCY
- PROBABILITY : BASIC NOTIONS.

EXPERIMENTS (MEYER 1970)

E1: Toss a die and observe the number that shows on top.

E2: Toss a coin 4 times and observe # heads.

E3: Toss a coin 4 times and observe the sequence of heads and tails obtained.

E4: Manufacture items on a production line and count # defective items produced during a 24-hour period.

E5: An airplane wing is manufactured and # of defective rivets are counted.

E6: A light bulb is manufactured and is tested for how many hours it will last.

E7: A lot of 10 items contain 3 defective items. One item is chosen after another, without replacement, until the last defective item is obtained. The total number of items that are chosen are counted.

E8: Items are manufactured until 10 non defective items are produced. The total number of manufactured items are counted.

E9: A rocket is launched and at a specific time t, its 3 velocity components v_x, v_y and v_z recorded.

E10: A rocket is launched and at times $t_1, t_2, \ldots, t_n$ the rockets height above the ground is recorded.

E_{11}: Tensile strength of a steel beam is measured.

E_{12}: From an urn containing only black balls, a ball is chosen and its color noted.

E_{13}: Temperature is recorded at a specific location at a specific time.

E_{14}: In the 24-hour period the min & max temp. are recorded at a specific location.

Observations

1 Each experiment is capable of being repeated indefinitely under essentially unchanged condition.

2. We may not be able to state what a particular outcome will be, but we are able to describe the set of all possible outcomes.

3. As an experiment is performed repeatedly, the individual outcomes occur in a haphazard manner. However, if you repeat the experiment a large number of times, we may be able to construct a mathematical model with which to analyze the experiment.
e.g. When tossing a coin, we don't know whether the next toss will be Head or Tail, but we know that if we observe a large number of tosses, we obtain almost same # heads as # tails

4. E_{12} is a useless experiment.

Sample Space

With each experiment, the sample space is the set of all possible outcomes of that experiment.

For our examples the sample spaces are

$S_1 = \{1, 2, 3, 4, 5, 6\}$

$S_3 = \{$all possible sequences $a_1 a_2 a_3 a_4$ where each $a_i \in \{H, T\}\}$

$S_2 = \{0, 1, 2, 3, 4\}$

$S_4 = \{0, 1, 2, 3, \ldots, N\}$ where $N =$ max # items that can be produced in 24-hours.

$S_5 = \{0, 1, 2, 3, \ldots, M\}$ where $M =$ max # rivets.

$S_6 = \{t \mid t \geq 0\}$

$S_7 = \{3, 4, 5, 6, 7, 8, 9, 10\}$

$S_8 = \{10, 11, 12, 13, \ldots\}$

$S_9 = \{(v_x, v_y, v_z) \mid v_x, v_y, v_z \in \mathbb{R}\}$

$S_{10} = \{(h_1, h_2, \ldots, h_n) \mid h_i \geq 0\}$

$S_{11} = \{T \mid T \geq 0\}$ (I don't know the units!)

$S_{12} = \{$Black Ball$\}$

$S_{13} = ??$ $\{f \mid f$ is a differentiable function s.t $m \leq f(t) \leq M$ for all t, where m and M are the min & max temp.$\}$.

$S_{14} = \{(x, y) \mid m \leq x \leq y \leq M\}$

Some Observations

1. We should talk about sample space associated with an experiment

$$S2 = \{0, 1, 2, 3, 4\} \; \& \; S3 = \{HHHH, THHH, \cdots\}$$

2. Out come of an experiment need not be a number

See $S3$, $S9$ & $S10$ are vectors, $S13$ is a function

3. Number of out comes in a sample space
   - finite ($S_1, S_2, S_3, S_4, S_5, S_7, S_{12}$)
   - countably infinite (S_8)
   - uncountable ($S_6, S_9, S_{10}, S_{11}, S_{13}, S_{14}$)

Consider $E_6 \equiv$ A light bulb is tested for how many hours it will last.

Since we can only measure time with some accuracy, and for the bulb, it doesn't have to be so accurate, we can assume that the time is descrete (say minimum accuracy is 1 minute). Moreover we assume that the bulb can last only at most H hours. Thus the sample space then becomes finite.

$$\{1min, 2min, \ldots, 60min, \cdots, 60H\}.$$

Lesson: Choose the sample space that is convenient to work with.

EVENTS

An event A, with respect to a particular sample space S associated with an Experiment $\mathcal{E}$, is a set of possible outcomes, i.e $A \subseteq S$.

$\Rightarrow$ S and ϕ are events as $S \subseteq S$ and $\phi \subseteq S$.
Also, individual outcomes are events.

A_1: An even number occurs $A_1 = \{2, 4, 6\}$

A2: Two heads occurs $A_2 = \{2\}$

A3: More heads than tails $A_3 = \{HHHH, HHHT, HHTH, HTHH, THHH\}$

A4: All items are non-defective $A_4 = \{0\}$.

A5: More than 5 were defective $A_5 = \{6, 7, 8, \ldots, M\}$

A6: Bulb lasts for at most 6 hours $A_6 = \{t \mid t \leq 6\}$.

A14: Max is 10 degrees higher than Min
$A_{14} = \{(x, y) \mid y = x + 20\}$

Since events are subsets of S, we observe following

1. If A and B are events then so is

   $A \cup B$: It occurs iff either A occurs or B occurs or both occurs

   $A \cap B$: It occurs iff both A and B occurs

2. If A is an event, then $\bar{A}$ is an event.

   $\bar{A}$ occurs if and only if A does not occur.

3. Similarly, if $A_1, A_2, \ldots, A_n$ are finite collection of events then

   $\bigcup_{i=1}^{n} A_i$ occurs iff at least one of events A_i occurs

   $\bigcap_{i=1}^{n} A_i$ occurs iff all the events A_i occur.

4. Consider n repetitions of an experiment $\mathcal{E}$ whose sample space is S.

   $S \times S \times \cdots \times S = \{(s_1, s_2, \ldots, s_n) \mid s_i \in S, i = 1..n\}$. represents the set of all possible outcomes.

   e.g. Tossing a coin n-times

   $$S \times S \times \cdots \times S = \{(s_1, s_2, \ldots, s_n) \mid s_i \in \{H, T\}, i = 1..n\}$$

   an event A may be that the first 3 tosses are heads and rest can be anything.

   then $A = \{(s_1, s_2, s_3, \ldots, s_n) \mid s_1 = H, s_2 = H, s_3 = H, s_i \in \{H, T\} \text{ for } 3 \le i \le n\} \subseteq S \times S \times \cdots \times S$.

Mutually Exclusive Events.

Two events A and B are said to be mutually exclusive if they cannot occur together.

$A \cap B = \phi$

Example: A new drink is launched in the market and a survey is done to record how many people like the drink.

Sample Space $S = \{ t \mid t \geq 0 \text{ and } t \text{ is a number} \}$

Consider three events

$A = \{ t \mid t < 100 \}$ $\quad B = \{ t \mid 50 \leq t \leq 200 \}$

$C = \{ t \mid t > 150 \}$.

Then $A \cup B = \{ t \mid 0 \leq t \leq 200 \}$

$A \cap B = \{ t \mid 50 \leq t < 100 \}$

$B \cup C = \{ t \mid t \geq 50 \}$

$B \cap C = \{ t \mid 150 < t \leq 200 \}$

$A \cap C = \phi$ $\leftarrow$ MUTUALLY EXCLUSIVE

$A \cup C = \{ t \mid 0 \leq t < 100 \text{ or } t > 150 \}$

$\bar{A} = \{ t \mid t \geq 100 \}$

$\bar{C} = \{ t \mid 0 \leq t \leq 150 \}$

Relative Frequency

Let A be an event associated with an experiment

e.g. Roll a die and let A be the event of outcome be an even number.

Experiment

$S = \{1,2,3,4,5,6\}$ and $A = \{2,4,6\}$.

We cannot say for certainity that when we do this experiment, whether A will or will not occur.

Therefore, we associate a number with A which says how LIKELY A will occur.

Suppose we repeat an experiment $\mathcal{E}$, n-times and let the event A occurs n_A times.

Define $f_A = n_A/n$ as the relative frequency of A

Note that

(a) $0 \le f_A \le 1$

(b) $f_A = 1$ iff A occurs each time among n-trials

(c) $f_A = 0$ iff A never occurs in the n-trials

(d) If A & B are mutually exclusive events then $f_{A \cup B} = f_A + f_B$

$$\boxed{\text{As } n \to \infty, \quad f_A \to Pr(A), \text{ i.e } f_A \text{ converges to a value } Pr(A)}$$

Consider a coin, and start tossing it and keep track of outcomes.

Here is an actual toss sequence I did

H H T T H T H H T T T H . . . .

Let A be the event of obtaining "Heads".

Let us compute f_A after each toss

$n=1 \quad f_A = 1/1 = 1$

$n=2 \quad f_A = 2/2 = 1$

$n=3 \quad f_A = 2/3 = 0.66$

$n=4 \quad f_A = 2/4 = 0.5$

$n=5 \quad f_A = 3/5 = 0.6$

$n=6 \quad f_A = 3/6 = 0.5$

$n=7 \quad f_A = 4/7 = 0.57$

$n=8 \quad f_A = 5/8 = 0.63$

$n=9 \quad f_A = 5/9 = 0.55$

$n=10 \quad f_A = 5/10 = 0.5$

$n=11 \quad f_A = 5/11 = 0.45$

$n=12 \quad f_A = 6/12 = 0.5$

As $n \to \infty$

$f_A \to 0.5 = Pr(A)$.

As n is increased, the relative frequency stablizes, though we cannot say what the next outcome will be with certainity.