Probability

Sample Space S corresponding to Experiment $\mathcal{E}$.

We assume S is finite.

Probability Function $Pr : S \to \mathbb{R}^+$

such that

(1) $\forall \omega \in S, \quad 0 \leq Pr(\omega) \leq 1$

(2) $\sum_{\omega \in S} Pr(\omega) = 1$

For an Event $A \subseteq S$

$$Pr(A) = \sum_{\omega \in A} Pr(\omega)$$

Probability

Let $\mathcal{E}$ be an experiment. Let S be a sample space associated with $\mathcal{E}$. With each event A, we associate a real number $Pr(A)$, called probabaity of A, that satisfies following properties:

P1. $0 \leq Pr(A) \leq 1$

P2. $Pr(S) = 1$

P3. If A and B are mutually exclusive

$$Pr(A \cup B) = Pr(A) + Pr(B).$$

P4. If $A_1, A_2, \ldots, A_n$ are pairwise mutually exclusive events then $Pr\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} Pr(A_i)$

→ Claim: $Pr(\phi) = 0$.

Pf: Note that empty set is an event.

Since $A = A \cup \phi$, and A and ϕ are mutually exclusive,

by P3:

$$Pr(A) = Pr(A \cup \phi) = Pr(A) + Pr(\phi) \Rightarrow Pr(\phi) = 0.$$

Question: Suppose A is an event and given that $Pr(A) = 0$. Does it mean that $A = \phi$?

<u>Claim 2</u>: Let A be an event, then

$$Pr(A) = 1 - Pr(\bar{A})$$

Proof: Sample Space $S = A \cup \bar{A}$

and A and $\bar{A}$ are mutually exclusive.

P2 & P3 give $Pr(S) = 1 = Pr(A \cup \bar{A})$

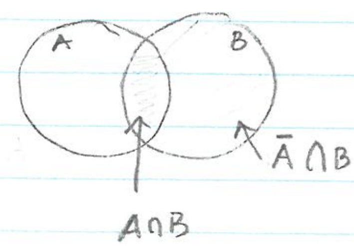
$$= Pr(A) + Pr(\bar{A}). \quad \blacksquare$$

<u>Claim 3</u>: If A and B are two events, then

$$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$$

Pf: $A \cup B = A \cup (B \cap \bar{A})$

$B = (A \cap B) \cup (B \cap \bar{A})$



$\underset{\text{by P3}}{\Longrightarrow}$ $Pr(A \cup B) = Pr(A) + Pr(B \cap \bar{A})$

$Pr(B) = Pr(A \cap B) + Pr(B \cap \bar{A})$

Hence,

$$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B) \quad \blacksquare$$

<u>Claim 4</u>: If $A \subseteq B$, then $Pr(A) \leq Pr(B)$.

Proof: $B = A \cup (B \cap \bar{A})$. By P3: $Pr(B) = Pr(A) + Pr(B \cap \bar{A}) \geq Pr(A)$

as $Pr(B \cap \bar{A}) \geq 0$ by P1 $\blacksquare$

$Pr(A)$ and f_A

$f_A \equiv$ frequency of A in n-trials.

Note that f_A approximated $Pr(A)$, but is not the same as $Pr(A)$.

What is $Pr(A)$?

- We have associated a real number with an event A, that satisfies Properties P1–P4. These numbers satisfy certain claims (Claims 1–4). Now how do we actually obtain the numerical value of of $Pr(A)$? has nothing to do with validity of these claims.

- f_A may or may not approximate $Pr(A)$ well, but that does not change anything about P1–P4 or Claims 1–4.

Examples

1. COIN

Flip a coin and outcome is either a Head or Tail.

$\Rightarrow$ Sample Space $S = \{H, T\}$

If the coin is fair then the probability of head is same as probability of Tail, i.e. $Pr(H) = Pr(T)$.

Since $Pr : S \to \mathbb{R}^+$; $Pr(H) = Pr(T) = 1/2$.

Note that they are positive and add up to 1.

Possible events are $\phi, \{H\}, \{T\}, \{H,T\}$.

$Pr(\phi) = 0$	i.e, if $A \subseteq S$ is an event
$Pr(\{H\}) = 1/2$	then $Pr(A) = \sum_{\omega \in A} Pr(\omega)$
$Pr(\{T\}) = 1/2$	(this is the probability of event A).
$Pr\{\{H,T\}) = 1$	Note that $S \subseteq S$, thus $Pr(S) = \sum_{\omega \in S} Pr(\omega) = 1$.

Now suppose we flip the coin twice, the 4 outcomes are

$S = \{HH, HT, TH, TT\}$.

HT : First flip results in H and
Second flip results in T.

If the coin is fair, probability function

$Pr: S \to \mathbb{R}^+$ is given by

$$Pr(HH) = Pr(HT) = Pr(TH) = Pr(TT) = 1/4$$

Each element of sample space $S = \{HH, HT, TH, TT\}$ has positive probability and they add up to 1.

possible events $= 2^4 = 16$ as $|S| = 4$.

Consider the event $A = \{HT, TH\} \subseteq S$.

$Pr(A) = Pr(HT) + Pr(TH) = 1/4 + 1/4 = 1/2$.

= Probability of seeing exactly one head and one tail.

2. DICE: Roll a fair die twice.

Sample $S = \{(i, j) \mid 1 \le i \le 6, 1 \le j \le 6\}$

outcome of 1st die   outcome of 2nd die

$|S| = 36$, and since each outcome (i, j) has same probability, the probability of each outcome will be $1/36$.

(Note that it satisfies the requirement of probability function $Pr: S \to \mathbb{R}^+$)

Consider the following events

A_1 = The two rolls show the same
$= \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$: $Pr(A_1) = 6/36 = 1/6$.

A_2 = The sum is 7 or 11

There are six outcomes of the form $(i, 7-i)$ that sum to 7. Namely $\{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

and two outcomes $(5,6)$ and $(6,5)$ that add up to 11.

$$\Rightarrow Pr(A_2) = 8/36 \equiv 2/9 .$$

A_3 = they have no common factor greater than unity.

First, list all events that have factor greater than unity. They include

$\{(i,i) : i = 2,3,4,5,6\}$, $(2,4)$ $(2,6)$, $(3,6)$, $(4,2)$, $(4,6)$, $(6,2)$, $(6,3)$ $(6,4)$

In all there are 13 such events,

i.e. $Pr(\bar{A_3}) = 13/36$

or $Pr(A_3) = 1 - Pr(\bar{A_3}) = 1 - 13/36 \equiv 23/36.$

A_4 = First roll shows a number smaller than the second roll

All events of type $\{(i,j) \mid i<j\}$

$A_4 \equiv \{(1,2), (1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6)\}$

Thus $Pr(A_4) = 15/36$

$A_5 = \{ (i,j) \in S : i+j \text{ is even} \}$

$i+j$ is even if and only if

both i & j are even

or

both i & j are odd

Let A' be the event that both i & j are even

$A' = \{ (i,j) | \; i \text{ is even and } j \text{ is even} \}$

$Pr(A') = 3 \cdot 3/36 = 9/36$

Let A'' be the event that both i & j are odd.

$A'' = \{ (i,j) | \; i \text{ is odd and } j \text{ is odd} \}$

$Pr(A'') = 3 \cdot 3/36 = 9/36.$

$Pr(A_5) = Pr(A') + Pr(A'')$ as A' and A'' are disjoint events.

$= 9/36 + 9/36 = 18/36 = 1/2.$

$A_6 = \{ (i,j) \in S : i+j \text{ is odd} \}.$

Since the sum $i+j$ is either even or odd,

thus $A_6 = 1 - \bar{A_5} = 1 - 1/2 = 1/2.$

Example 2:

Let $S = \{1, 2, 3, 4, \ldots, 1000\}$

Assume each element has ~~the~~ equal probability of being chosen.

What is the probability that a number x chosen from S is divisible by 2 or 3.

Define two events A and B, where

$A = \{i \in S : i \text{ is divisible by } 2\}$

$B = \{j \in S : j \text{ is divisible by } 3\}$

$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$

$Pr(A) = 500/1000$ since $\left\lfloor \frac{1000}{2} \right\rfloor$ numbers are divisible by 2.

$Pr(B) = 333/1000$ since $\left\lfloor \frac{1000}{3} \right\rfloor$ numbers are divisible by 3.

$Pr(A \cap B) = 166/1000$ since $\left\lfloor \frac{1000}{6} \right\rfloor$ numbers are divisible by 6

$\Rightarrow Pr(A \cup B) = \dfrac{500 + 333 - 166}{1000} = \cancel{6667}\ 667/1000$

Example 3:

Assume there are equal chance of a newborn to be a boy or a girl. Consider the following scenario of a family

A1: The family wants to have 3 children

$B_i \equiv$ Event that i boys are born

$C \equiv$ more girls than boys.

Note that the sample space consists of outcomes each with equal probability

$$\{BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG\} \quad — \quad (*)$$

$Pr(B_1) \equiv 3/8 \qquad Pr(B_2) \equiv 3/8 \qquad Pr(C) \equiv 4/8 = 1/2$

A2: have children until the first girl is born or until three are born, whichever occurs first and then STOP

Outcomes are $\{\underset{F_1}{G}, \underset{F_2}{BG}, \underset{F_3}{BBG}, \underset{F_4}{BBB}\}$

~~$Pr(B_1) \equiv$~~

Caution: They may not have same probability.

Consider the sample space in (*)

F_2 corresponds to $\{BGG \cup BGB\}$. Thus ~~$Pr(B_2) = 2/8 = 1/4$~~

$\Rightarrow Pr(B_1) \equiv 2/8 = 1/4.$

F_1 corresponds to $\{GGG \cup GGB \cup GBG \cup GBB\}$

$\Rightarrow Pr(C) \equiv 4/8 = 1/2$

Some of the useful rules to compute probabilities

1. $Pr(\phi) = 0$

2. $A_1, A_2, A_3, \ldots, A_n$ are pairwise disjoint events, then
$$Pr(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} Pr(A_i)$$

3. For any event A, $Pr(A) \equiv 1 - Pr(\bar{A})$

4. If A and B are events $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$.

5. UNION BOUND
If $A_1, A_2, \ldots, A_n$ are a sequence of events, then
$$Pr(A_1 \cup A_2 \cup \cdots \cup A_n) \leq \sum_{i=1}^{n} Pr(A_i)$$

6. A & B are events and $A \subseteq B$ then $Pr(A) \leq Pr(B)$