Conditional Probability

Example: Consider that a manufacturer produces 100 items of which 20 are defective. Suppose we choose 2 items from the lot in the following two ways:

Method 1: We choose the first item, put it back in the lot and then choose second
(with replacement)

Method 2: We choose the first item, do not put it back, then choose the second.
(without replacement)

Define the events.

$A = \{\text{the first item is defective}\}$

$B = \{\text{the second item is defective}\}$

For Method 1 (with replacement)

$$Pr(A) = Pr(B) = \frac{20}{100} = \frac{1}{5}.$$

For Method 2 (without replacement)

$$Pr(A) = \frac{20}{100} = \frac{1}{5}$$

What about $Pr(B) = ?$

That depends on what was the first item?
If first item was non-defective $Pr(B) \to \frac{20}{99}$
If first item was defective $Pr(B) \to 19/99$

Define

$Pr(B|A) \equiv$ Conditional probability of the event B given that A has occured.

Thus

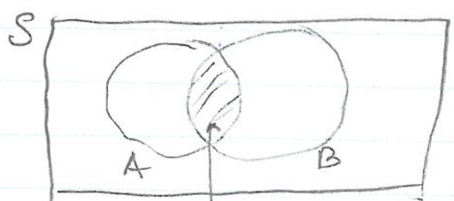
$$Pr(B|A) = \frac{19}{99}$$

19 → Only 19 defective items are left.

99 → 99 items are left after first item is removed

$$Pr(B|\bar{A}) = \frac{20}{99}.$$

Note: $Pr(B|A)$ — We are computing $Pr(B)$ with respect to a reduced sample space A, rather than with respect to full sample space S.



how probable we are in B, given that we are in A.

Example 2

Two fair dice are tossed.
Let x_1 be outcome of first.

Let x_2 be the outcome of second

Define two events

$A = \{(x_1, x_2) \mid x_1 + x_2 = 10\}$ $B = \{(x_1, x_2) \mid x_1 > x_2\}$

Note that

$Pr(A) = \frac{3}{36}$ $\leftarrow \{(5,5), (4,6), (6,4)\}$

$Pr(B) = \frac{15}{36}$ $\leftarrow \{(2,1), (3,1), (4,1), (5,1), (6,1), \ldots\}$

$Pr(A|B) = \frac{1}{15}$ $\leftarrow \{(6,4)\}$ from $\{(2,1), (3,1), \ldots\}$

$Pr(B|A) = 1/3$ $\leftarrow \{(6,4)\}$ from $\{(5,5), (4,6), (6,4)\}$

Let us look at $Pr(A \cap B) = \frac{1}{36}$.

Now observe that

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)} = \frac{1/36}{15/36} = 1/15$$

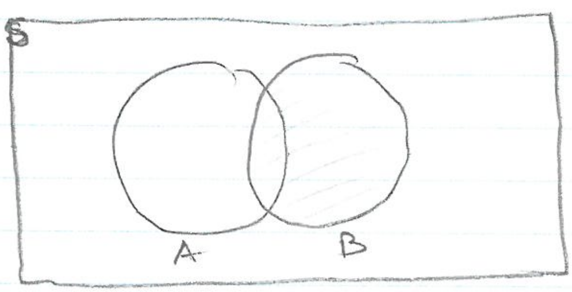
$$Pr(B|A) = \frac{Pr(A \cap B)}{Pr(A)} = \frac{1/36}{3/36} = 1/3$$

Definition

Let (S, Pr) be a probability space, and let A and B be two events with $Pr(B) > 0$.

The conditional probability $Pr(A|B)$ is defined as

$$\boxed{Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}}$$



→ Given B, event A occurs if only if $A \cap B$ occurs.

→ If both $Pr(A) > 0$, $Pr(B) > 0$, then $Pr(B|A) = \frac{Pr(A \cap B)}{Pr(A)}$.

→ Note that from the definition we can derive that $Pr(A \cap B) = Pr(A|B) \cdot Pr(B)$.

$$= Pr(B|A) \cdot Pr(A)$$

For example, consider Method 2, and we ask what is the probability that both items are defiective, i.e what is $Pr(A \cap B) = ?$

$$Pr(A \cap B) = Pr(A|B) \cdot Pr(B) = Pr(B|A) \cdot Pr(A)$$

$$= \frac{19}{99} \cdot \frac{20}{100}$$

$$= 19/495$$