Some of the useful rules to compute probabilities

1. $Pr(\phi) = 0$

2. $A_1, A_2, A_3, \ldots, A_n$ are pairwise disjoint events, then

$$Pr(A_1 \cup A_2 \cup \cdots \cup A_n) \equiv \sum_{i=1}^{n} Pr(A_i)$$

3. For any event A, $Pr(A) \equiv 1 - Pr(\bar{A})$

4. If A and B are events $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$

5. UNION BOUND

If $A_1, A_2, \ldots, A_n$ are a sequence of events, then

$$Pr(A_1 \cup A_2 \cup \cdots \cup A_n) \leq \sum_{i=1}^{n} Pr(A_i)$$

6. A & B are events and $A \subseteq B$ then $Pr(A) \leq Pr(B)$

7. $$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)} \iff Pr(A \cap B) = Pr(A|B)\, Pr(B)$$

8. A and B are independent if $Pr(A \wedge B) = Pr(A)\, Pr(B)$

Conditional Probability

Example: Consider that a manufacturer produces 100 items of which 20 are defective. Suppose we choose 2 items from the lot in the following two ways:

Method 1: We choose the first item, put it back in the lot and then choose second (with replacement)

Method 2: We choose the first item, do not put it back, then choose the second. (without replacement)

Define the events.

$A = \{\text{the first item is defective}\}$

$B = \{\text{the second item is defective}\}$

For Method 1 (with replacement)

$$Pr(A) = Pr(B) = \frac{20}{100} = \frac{1}{5}.$$

For Method 2 (without replacement)

$$Pr(A) = \frac{20}{100} = \frac{1}{5}$$

What about $Pr(B) = ?$

That depends on what was the first item?

If first item was non-defective $Pr(B) \rightarrow \frac{20}{99}$

If first item was defective $Pr(B) \rightarrow 19/99$

Define

$Pr(B|A) \equiv$ Conditional probability of the event B given that A has occured.

Thus

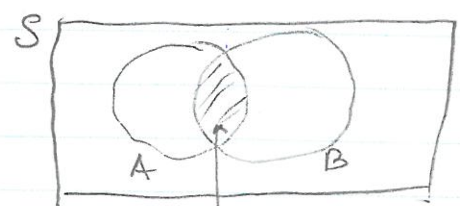
$$Pr(B|A) = \frac{19}{99}$$

19 → Only 19 defective items are left.

99 → 99 items are left after first item is removed

$$Pr(B|\bar{A}) = \frac{20}{99}.$$

Note: $Pr(B|A)$ — We are computing $Pr(B)$ with respect to a reduced sample space A, rather than with respect to full sample space S.



how probable are we are in B, given that we are in A.

Example 2

Two fair dice are tossed.
Let x_1 be outcome of first.

Let x_2 be the outcome of second.

Define two events

$A = \{(x_1, x_2) \mid x_1 + x_2 = 10\}$ $\quad B = \{(x_1, x_2) \mid x_1 > x_2\}$

Note that

$Pr(A) = \frac{3}{36}$ $\quad \leftarrow \{(5,5), (4,6), (6,4)\}$

$Pr(B) = \frac{15}{36}$ $\quad \leftarrow \{(2,1), (3,1), (4,1), (5,1), (6,1), \ldots\}$

$Pr(A|B) = \frac{1}{15}$ $\quad \leftarrow \{(6,4)\}$ from $\{(2,1), (3,1), \ldots\}$

$Pr(B|A) = 1/3$ $\quad \leftarrow \{(6,4)\}$ from $\{(5,5), (4,6), (6,4)\}$

Let us look at $Pr(A \cap B) = \frac{1}{36}$.

Now observe that

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)} = \frac{1/36}{15/36} = 1/15$$

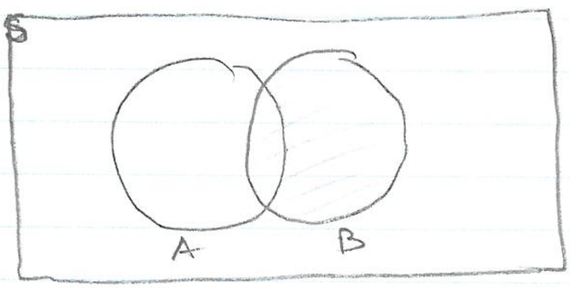
$$Pr(B|A) = \frac{Pr(A \cap B)}{Pr(A)} = \frac{1/36}{3/36} = 1/3$$

Definition

Let (S, Pr) be a probability space, and let A and B be two events with $Pr(B) > 0$.

The conditional probability $Pr(A|B)$ is defined as

$$\boxed{Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}}$$



→ Given B, event A occurs if only if $A \cap B$ occurs.

→ If both $Pr(A) > 0$, $Pr(B) > 0$, then $Pr(B|A) = \dfrac{Pr(A \cap B)}{Pr(A)}$.

→ Note that from the definition we can derive that $Pr(A \cap B) = Pr(A|B) \cdot Pr(B)$.
$= Pr(B|A) \cdot Pr(A)$

For example, consider Method 2, and we ask what is the probability that both items are defective, i.e what is $Pr(A \cap B) = ?$

$$Pr(A \cap B) = Pr(A|B) \cdot Pr(B) = Pr(B|A) \cdot Pr(A)$$

$$= \frac{19}{99} \cdot \frac{20}{100}$$

$$= 19/495$$

Claim: Let A and B be two events with $Pr(B) > 0$.

Then $Pr(A|B) + Pr(\bar{A}|B) = 1$

Proof: $Pr(A|B) + Pr(\bar{A}|B)$

$$= \frac{Pr(A \cap B)}{Pr(B)} + \frac{Pr(\bar{A} \cap B)}{Pr(B)} \quad \text{— (1)}$$

Events $A \cap B$ and $\bar{A} \cap B$ are disjoint

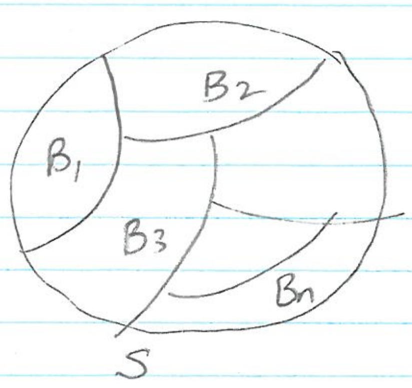
thus $Pr(A \cap B) + Pr(\bar{A} \cap B) = Pr(A \cap B \cup \bar{A} \cap B)$

$= Pr(B)$

Thus (1) is same as

$$= \frac{Pr(B)}{Pr(B)} = 1. \quad \blacksquare$$

Next up: Law of Total Probility.



Assume that $B_1, B_2, \ldots, B_n$ is a partition of S

i.e

(a) $B_1 \cup B_2 \cup \cdots \cup B_n = S$

(b) $\forall i, j, i \neq j, \quad B_i \cap B_j = \phi$.

Theorem [Law of total probability]

Let (S, Pr) be a probability space and let A be an event.

Assume that $B_1, B_2, ---, B_n$ be events such that

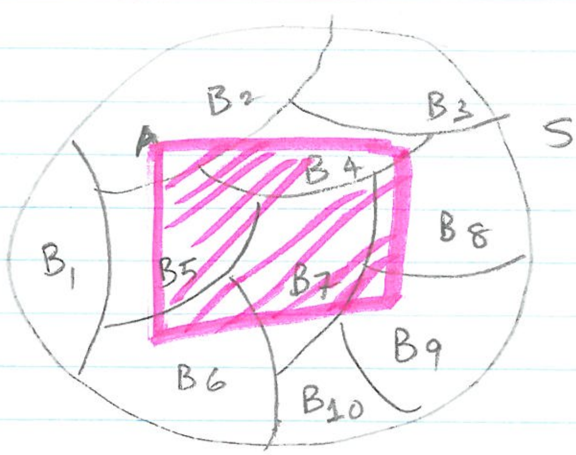
1) $Pr(B_i) > 0 \quad \forall i, \quad 1 \le i \le n$

2) the events $B_1, B_2, ---, B_n$ are pairwise disjoint.

3) $\bigcup_{i=1}^{n} B_i = S$

Then

$$Pr(A) = \sum_{i=1}^{n} Pr(A|B_i) \cdot Pr(B_i)$$



Proof: $A = A \cap S$ as $A \subseteq S$.

$$= A \cap \left(\bigcup_{i=1}^{n} B_i \right)$$

$$= \bigcup_{i=1}^{n} (A \cap B_i)$$

Since $B_1, B_2, ---, B_n$ are pairwise disjoint, then $A \cap B_1, A \cap B_2, ---, A \cap B_n$ are pairwise disjoint.

Thus,

$$Pr(A) = Pr\left(\bigcup_{i=1}^{n} A \cap B_i \right)$$

$$= \sum_{i=1}^{n} Pr(A \cap B_i)$$

$$= \sum_{i=1}^{n} Pr(A|B_i)\, Pr(B_i)$$

as $Pr(A|B_i) = \dfrac{Pr(A \cap B_i)}{Pr(B_i)}$

by definition of conditional probability ☒

Applications

Example 1:

Experiment:

Flip a coin $\xrightarrow{\text{Heads}}$ Roll a fair die and let R be result

Flip a coin $\xrightarrow{\text{Tails}}$ Roll two fair dice. Let R be the sum.

Define the event

$A = \{\text{the value of } R = 2\}$

Problem: Compute $\Pr(A) = ?$

Define the event $B = \{\text{coin comes up heads}\}$.

Note ① $\Pr(B) > 0$, $\Pr(\bar{B}) > 0$

② $B \cap \bar{B} = \phi$

③ $S = B \cup \bar{B}$ = possible outcomes of toss

Then,

$$\Pr(A) = \Pr(A|B)\Pr(B) + \Pr(A|\bar{B})\Pr(\bar{B})$$

$\Pr(A|B)$ = given that Toss is head then what is the probability we get a 2.

$= 1/6$

$$\Pr(A|\bar{B}) = 1/36$$

Thus $\Pr(A) = 1/6 \cdot 1/2 + 1/36 \cdot 1/2 = 7/72$ ∎

Applications

Example 2: Factory producing 100 items, 20 Defective, 80 non-defective. We choose two items without replacement.

Events $I = \{\text{first chosen item is defective}\}$

$II = \{\text{Second chosen item is defective}\}$.

Suppose we want to compute $Pr(II) = ?$

Let $B_1 = I$ and $B_2 = \bar{I}$

Note that $S = I \cup \bar{I}$ and $I \cap \bar{I} = \phi$ and $Pr(I) > 0$, $Pr(\bar{I}) > 0$

Let $B_1 = I$, $B_2 = \bar{I}$, and $A = II$. and apply theorem

$$Pr(II) = Pr(A) = \sum_{i=1}^{2} Pr(A|B_i) \cdot Pr(B_i)$$

$$= Pr(A|B_1) \cdot Pr(B_1) + Pr(A|B_2) \cdot Pr(B_2)$$

$$= \frac{19}{99} \cdot \frac{1}{5} + \frac{20}{99} \cdot \frac{4}{5}$$

$$= \frac{1}{5}.$$

(Probability that the second item is defective is $1/5$ under without replacement).

Example 3: An item is produced by 3-factories

F_1 produces $2x$ items

F_2, F_3 produce x items each.

	F_1	F_2	F_3
# items	$2x$	x	x
% defective	2%	2%	4%

All items are put in a pile and one is drawn at random.

Question: Pr (drawn item is defective) = ?

$A = \{\text{drawn item is defective}\}$

$B_1 = \{\text{drawn item is from } F_1\}$

$B_2 = \{\text{———————— } F_2\}$

$B_3 = \{\text{———————— } F_3\}$

Note that ① $Pr(B_1) > 0$, $Pr(B_2) > 0$, $Pr(B_3) > 0$

② $B_i \cap B_j = \phi \quad i \neq j \quad i, j \in \{1,2,3\}$

③ $B_1 \cup B_2 \cup B_3 = S$ are all possibilities for the drawn item.

Thus

$$Pr(A) = Pr(A|B_1) Pr(B_1) + Pr(A|B_2) \cdot Pr(B_2) + Pr(A|B_3) \cdot Pr(B_3)$$

$$= 0.02 * \frac{1}{2} + 0.02 * \frac{1}{4} + 0.04 * \frac{1}{4}$$

$$= 0.025 \blacksquare$$

Question: Suppose that an item is chosen from the pile, and is found to be defective.

What is the probability that it is produced by F_1

$Pr(B_1/A) = ?$

BAYES THEOREM

Let $B_1, B_2, \ldots, B_n$ be partition of sample space S.
Let $A \subseteq S$ be an event.

$$Pr(B_i/A) = \frac{Pr(A|B_i)\,Pr(B_i)}{\sum_{j=1}^{n} Pr(A|B_j)\,Pr(B_j)}$$

$$Pr(B_1|A) = \frac{Pr(A|B_1) \cdot Pr(B_1)}{Pr(A|B_1)\,Pr(B_1) + Pr(A|B_2) \cdot Pr(B_2) + Pr(A|B_3) \cdot Pr(B_3)}$$

$$= \frac{0.02 * 1/2}{0.02 * 1/2 + 0.02 * 1/4 + 0.04 * 1/4}$$

$$= 0.40$$

Note: Bayes thm allows us to write conditional probabilities in terms of similar conditional probabilities but the conditions are rereversed.

Two events A are B are said to be independent if $Pr(A \cap B) = Pr(A)\,Pr(B)$.

Example: Roll a fair die twice.

$A = \{$the first roll shows an even number$\}$

$B = \{$2nd Roll shows 5 or 6$\}$

$$Pr(A) = 1/2 \qquad Pr(B) = 1/3 \qquad Pr(A \cap B) = ? = 1/6$$

$$= Pr(A)\,Pr(B)$$

Example 2: Roll two fair die

$A = \{$the first die shows even number$\}$

$B = \{$the second dies shows an odd number$\}$.

$C = \{$two die shows both odd or both even numbers$\}$

Observe $Pr(A) = Pr(B) = Pr(C) = 1/2$

$$\left.\begin{array}{l} Pr(A \cap B) = 1/4 = Pr(A)\,Pr(B) \\ Pr(A \cap C) = 1/4 = Pr(A)\,Pr(C) \\ Pr(B \cap C) = 1/4 = Pr(B)\,Pr(C) \end{array}\right\} \text{pairwise independent}$$

$$Pr(A \cap B \cap C) = 0 \neq Pr(A)\,Pr(B)\,Pr(C)$$

⇒ ~~Pair~~

Definition: 3 events A, B, C are mutually independent iff

$$Pr(A \cap B) = Pr(A)\,Pr(B)$$
$$Pr(B \cap C) = Pr(B)\,Pr(C)$$
$$Pr(A \cap C) = Pr(A)\,Pr(C)$$
$$Pr(A \cap B \cap C) = Pr(A)\,Pr(B)\,Pr(C)$$

Mutually Independent Events

Let (S, Pr) be a probability space.

A sequence of events $A_1, A_2, \ldots, A_n$; $n \geq 2$;
is said to be Mutually Independent Events if
for all k, $2 \leq k \leq n$ and all
$i_1 < i_2 \cdots < i_k$

$$Pr(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = Pr(A_{i_1}) \cdot Pr(A_{i_2}) \cdots \cdot Pr(A_{i_k})$$

As noted in previous example,
pairwise independence $\not\Rightarrow$ mutual independence.

Examples of mutual independence.

Question: Let $n \geq 1$ and we flip n-fair coins.

Define event $A_i = \{i^{th}$ coin comes up heads$\}$

for $1 \leq i \leq n$.

Define $A = A_1 \wedge A_2 \wedge A_3 \cdots \wedge A_n$, i.e all n-flips are heads.

Clearly $Pr(A) = 1/2^n$.

Alternatively, using independence, as each coin flip is independent of any other flip.

$$Pr(A) = Pr(A_1) \cdot Pr(A_2) \cdot \cdots \cdot Pr(A_n) = \left(\frac{1}{2}\right)^n = \frac{1}{2^n}.$$

Question:

Consider a circuit C that consists of n-components $C_1, C_2, \ldots, C_n$. Let p be a real number $0 < p < 1$.

Assume that any component may fail with probability p, and its failure is indep of other components

Define events $1 \leq i \leq n$, where

$A_i = \{$" Component C_i fails "$\}$

Suppose C is configured as

C fails if any of C_i fails.

$A = \{$"the circuit C fails"$\}$

$Pr(A) = ?$

Note that

$$A = A_1 \vee A_2 \vee A_3 ---- \vee A_n.$$

$$\Leftrightarrow \quad \bar{A} = \bar{A_1} \wedge \bar{A_2} \wedge \bar{A_3} --- \wedge \bar{A_n}$$

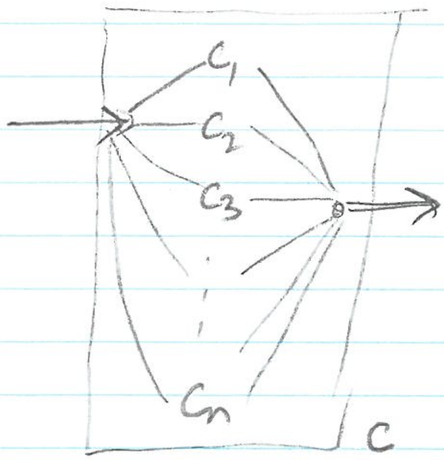
Thus

$$\begin{aligned} Pr(A) &= 1 - Pr(\bar{A}) \\ &= 1 - Pr(\bar{A_1} \wedge \bar{A_2} \wedge \bar{A_3} --- \wedge \bar{A_n}) \quad \text{Mutual Independence} \\ &= 1 - Pr(\bar{A_1})\, Pr(\bar{A_2}) --- Pr(\bar{A_n}) \\ &= 1 - (1-p)^n \end{aligned}$$

Thus for large values of n, $(1-p)^n \to 0$

$\Rightarrow Pr(A) \to 1$, i.e circuit will very likely fail.

Suppose C is configured as



Now C will fail only when all components fail.

Thus $A = A_1 \wedge A_2 \cdots \wedge A_n$

or $Pr(A) = Pr(A_1 \wedge A_2 \wedge \cdots \wedge A_n)$ Mutual Independence

$= Pr(A_1)\,Pr(A_2) \cdots Pr(A_n)$

$= p^n$

Thus, when $n \to \infty$ $p^n \to 0$,

$\Rightarrow$ it is unlikely the circuit will fail for large values of n.

5-12

How to choose a random element from a linked list

Given:

A link list L

$\rightarrow \square \rightarrow \square \rightarrow - - - - \rightarrow \square \rightarrow$ nil

- We do not know $|L|$ -
- We are only allowed to make a single pass

Problem: Choose a node of L, uniformly at random. i.e, if $|L| = n$, then each node has $pr = 1/n$ of being chosen.

```
Algorithm ChooseRandom(L):
u := head(L);
i := 1;
While u ≠ NIL do
  [ r := Random(i);      { Returns a random element with
                           uniform probability from {1 .. i} }
  [ if r = 1 then x := u;
  [ u := succ(u);
  [ i := i + 1
Return (x).
```

Observe

- In the first iteration $x =$ first element of L with $pr = 1$

- In the second iteration $x =$ second element of L with $pr = 1/2$, and it does not change with $pr = 1/2$ ~~$pr = 1/2$~~.

- In the third iteration $x =$ third element of L with $pr = 1/3$; value of x does not change with $pr = 2/3$.

⋮

- In the last iteration $x =$ last element of L with $pr = 1/|L|$; and its value does not change with $pr = |L|-1/|L|$.

To Show: Output of ChooseRandom(L) is a uniformly random node of List L. Let $|L| = n$, and let v be an arbitrary node of L. We will show, after the algorithm has terminated

$$X = V \text{ with } pr = \frac{1}{|L|} = \frac{1}{n}.$$

Pf: Let v be the k^{th} node of L, $1 \leq k \leq n$.

Note that $x = v$ if and only if

- during the k^{th} iteration, the value of x is set to v.
- for all $i = k+1, k+2, \ldots, n$; the value of x does not change in iteration i.

Define events

$A = \{$"After the algo has terminated $x = v$"$\}$.

For $1 \le i \le n$, define
$A_i = \{$" the value of x changes during the i^{th} iteration"$\}$

Then

$$A = A_k \wedge \overline{A_{k+1}} \wedge \overline{A_{k+2}} \wedge \cdots \wedge \overline{A_n}$$

Since events $A_k, A_{k+1}, \ldots, A_n$ are mutually independent.

$$Pr(A) = Pr\left(A_k \wedge \overline{A_{k+1}} \wedge \overline{A_{k+2}} \wedge \cdots \wedge \overline{A_n}\right)$$

$$= Pr(A_k)\, Pr(\overline{A_{k+1}})\, Pr(\overline{A_{k+2}}) \cdots Pr(\overline{A_n})$$

$$= \frac{1}{k}\left(1 - \frac{1}{k+1}\right)\left(1 - \frac{1}{k+2}\right) \cdots \left(1 - \frac{1}{n}\right)$$

$$= \frac{1}{k} \cdot \frac{k}{k+1} \cdot \frac{k+1}{k+2} \cdot \cdots \cdot \frac{n-1}{n}$$

$$= \frac{1}{n} \quad \blacksquare$$