COMP 2804

Discrete Structures II

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2016-03-09

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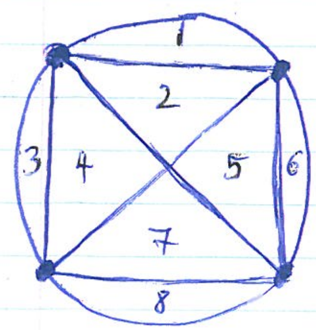
Topic: Cutting a Circle into Regions



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We are given a circle with n points on it.

How many regions do we obtain when connecting any pair of points with a line segment?

Today, you will learn ...

- an important lesson about deception.
- a geometric counting trick for plane graphs
- deriving a recurrence relation step-by-step
- assorted double-counting tricks
- so much stuff that will be on your exam!

Counting Regions when Cutting a Circle

Objectives

+ tackling a hard counting problem using recurrences

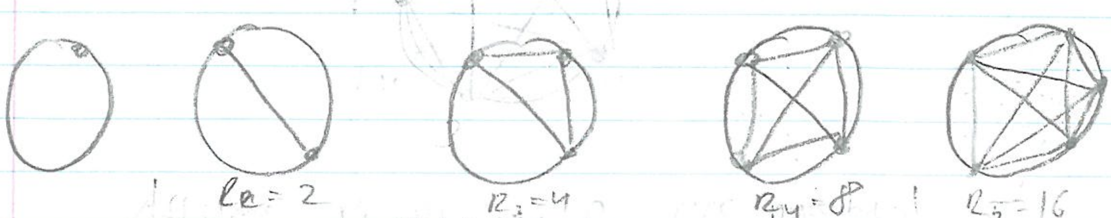
+ geometric counting strategies

+ small examples can be deceiving.

1) Problem Definition

- Place n points on a circle

- connect each pair of points with a straight line segment



- The segments divide the circle into R_n regions.

- We would like to determine R_n.

Deceiving Examples $R_6 = 31$

- from the examples: $R_1 = 1$, $R_2 = 2$, $R_3 = 4$, $R_4 = 8$, $R_5 = 16$

- It seems that $R_n = 2^{n-1}$ for all $n \geq 1$, but this guess turns out to be wrong.

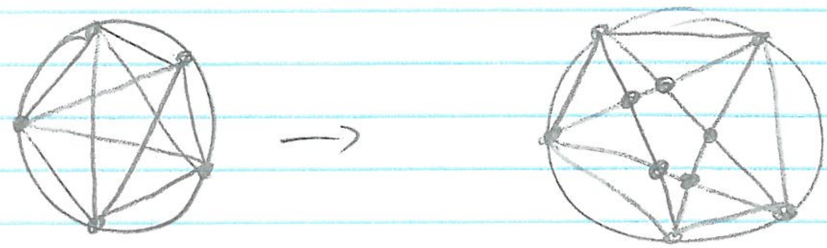
- we shall show that R_n grows polynomially, not exponentially in n.

2) A Polynomial Upper Bound on R_n.

We interpret the subdivided circle as a planar graph.

→ each point is a vertex
→ each intersection of two segments is a vertex
→ each arc or line segment becomes an edge.

for $n=5$



vertices = 5+5=10
#edges

$$\#\text{vertices} = \underset{\text{points}}{\underset{\uparrow}{5}} + \underset{\text{crossings}}{\underset{\uparrow}{5}} = 10$$

$$\#\text{edges} = \underbrace{5 + 3\cdot 5}_{\text{segments}} + \underset{\text{arcs}}{\underset{\uparrow}{5}} = 25$$

Let V_n = # vertices for n points and
let E_n = # edges — " — .

<u>claim 1)</u> $V_n \leq n + \binom{\binom{n}{2}}{2}$

Proof • There are n vertices on the circle.

• There are $\binom{n}{2}$ line segments connecting these points. Any of the $\binom{\binom{n}{2}}{2}$ pairs of line segments intersects at most once.

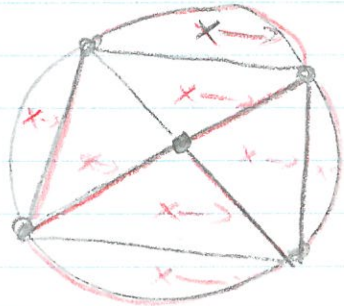
Not all line segments intersect, so
$$V_n \leq n + \binom{\binom{n}{2}}{2}.$$
□

claim2) $E_n \le n + \binom{V_n}{2}$

proof
- There are n circular arc edges along the circle. (double edges)
- Any other edge is a straight line edge connecting two of the V_n vertices.
  ↳ there are $\binom{V_n}{2}$ many pairs of vertices. □

claim 3) $R_n \le E_n$

We use a geometric counting argument.



- For each region r choose a point p_r inside r. (not at the same y-coordinate as any vertex)
- From p_r shoot a ray horizontally right, and let $f(r)$ be the first edge hit by that ray.
- Each edge is hit at most once, so f is injective

< coin analogy >

- Therefore, $R_n \le E_n$

□

We combine Claim 1, Claim 2, and Claim 3:

$$R_n \overset{\text{claim 3}}{\leq} E_n \overset{\text{claim 2}}{\leq} n + \binom{V_2}{2} \overset{\text{claim 1}}{\leq} n + \binom{n \ \binom{\binom{n}{2}}{2}}{2}$$

We use asymptotic notation to estimate the last term.

$$\binom{n}{2} = \frac{n(n-1)}{2} = O(n^2)$$

$$\Rightarrow \binom{\binom{n}{2}}{2} = \binom{O(n^2)}{2} = O(n^4)$$

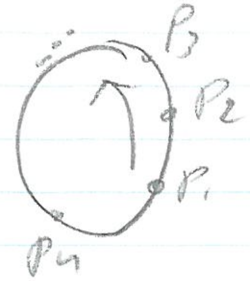
$$\Rightarrow n + \binom{\binom{n}{2}}{2} = n + O(n^4) = O(n^4)$$

$$\Rightarrow \binom{n + \binom{\binom{n}{2}}{2}}{2} = \binom{O(n^4)}{2} = O(n^8)$$

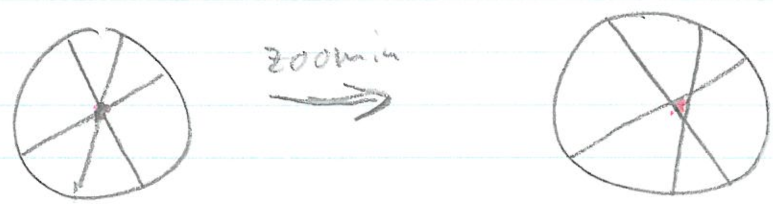
$$\Rightarrow R_n \leq n + \binom{n + \binom{\binom{n}{2}}{2}}{2} = n + O(n^8) = O(n^8)$$

- This proves that R_n grows polynomially in n.
- The bound $O(n^8)$ is not tight.
- In the remainder, we determine the exact value of R_n using recurrences and show $R_n = \Theta(n^4)$.

3) Recurrence for R_n



- We number the points $P_1, P_2, \ldots, P_n$ in counter-clockwise order along the circle
- No three line segments meet in a point



- We derive a recurrence for R_n as follows.

  → Remove all segments incident to P_n.
  ↳ now, we have R_{n-1} regions

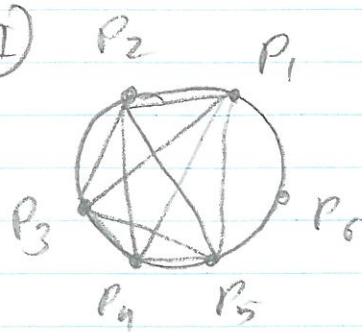
  → Add the line segments $P_1P_n, P_2P_n, \ldots, P_{n-1}P_n$ one by one. For each segment P_kP_n, let I_k be the increase in regions when adding P_kP_n.

$$R_n = R_{n-1} + \sum_{k=1}^{n-1} I_k \text{ for } n>1$$

with the base case $R_1 = 1$.

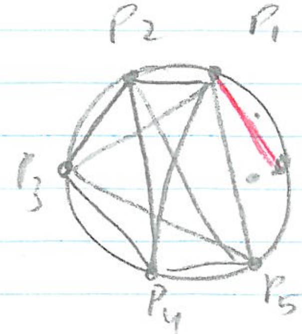
Example with $n = 6$

I



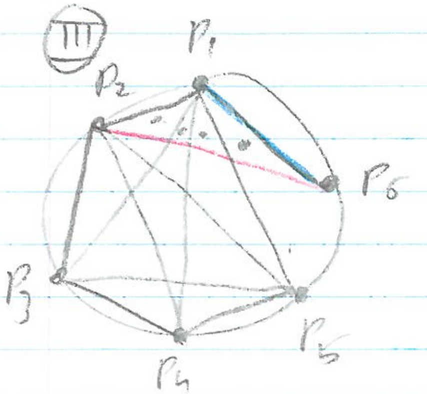
$R_5 = 16$

II



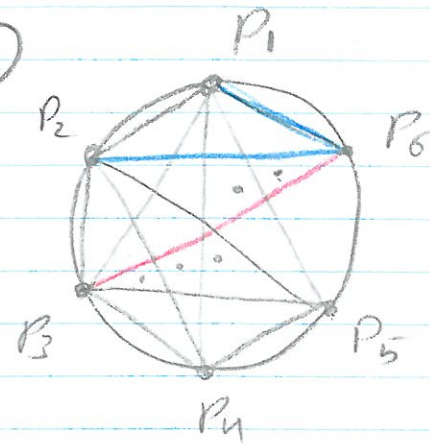
$I_1 = 1$

III



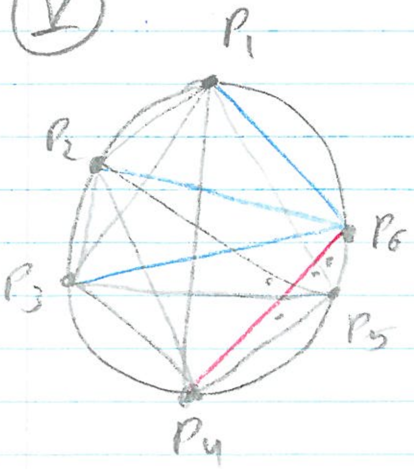
$I_2 = 4$

IV



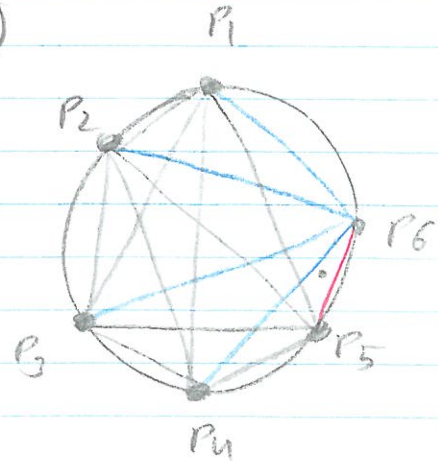
$I_3 = 5$

V



$I_4 = 4$

VI



$I_5 = 1$

$\hookrightarrow \quad R_6 = R_5 + I_1 + I_2 + I_3 + I_4 + I_5 = 16 + 1 + 4 + 5 + 4 + 1 = 31$

Abstracting from the Example

- Why was $I_3 = 5$? Because the segment p_3p_6 intersects the 4 segments $p_1p_4, p_1p_5, p_2p_4, p_2p_5$.

- I_3 = # new regions when adding p_3p_6

  $= (\underbrace{\text{\# intersections when adding}}_{= X = 4}\ p_3p_6) + 1 = 5$

- Any segment that intersects p_3p_6 has

  one endpoint above p_3p_6 and
  one endpoint below p_3p_6.

- Conversely, any pair of points a, b

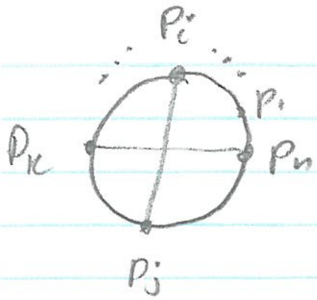
  with a above p_3p_6 and b below p_3p_6
  defines a segment ab that intersects p_3p_6.

- Two choices for a, namely p_1 and p_2,
  two choice for b, namely p_4, and p_5,
  then $X = 2 \cdot 2 = 4$.

Using these observations we derive the recurrence for the general case R_n, $n \geq 2$.

Recall

$$R_n = \underbrace{R_{n-1}}_{\text{\# regions when removing segments incident to } p_n} + \sum_{k=1}^{n-1} \underbrace{I_k}_{\text{\# regions added when reinserting segment } p_k p_n}$$



reinsert segment $p_k p_n$

When $i<j$, then

$p_i p_j$ intersects $p_k p_n$ if and only if

$1 \le i \le k-1$ and $k+1 \le j \le n-1$.

For a fixed k, there are

$k-1$ choices for i and
$n-k-1$ choices for j.

$\#$ intersections with $p_k p_n$ $= (k-1)(n-k-1)$

by the product rule. Therefore,

$$I_k = (k-1)(n-k-1)+1$$

We conclude

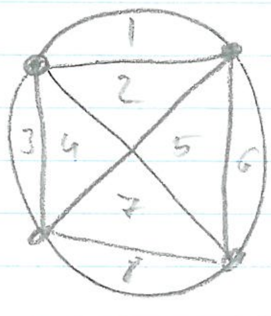
$$R_n = R_{n-1} + \sum_{k=1}^{n-1} I_k = R_{n-1} + \sum_{k=1}^{n-1} \left(1+(k-1)(n-k-1)\right)$$

$$= R_{n-1} + (n-1) + \sum_{k=1}^{n-1} (k-1)(n-k-1)$$

$$\Rightarrow \quad R_n = R_{n-1} + (n-1) + \sum_{k=2}^{n-2} (k-1)(n-k-1) \quad \text{for } n \ge 2$$

and $R_1 = 1$ is the desired recurrence.

2016-03-09 **Cutting a Circle into Regions**



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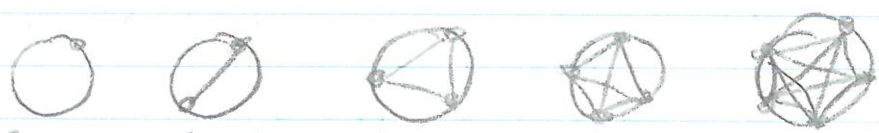
We are given a circle with n points on it.

How many regions do we obtain when connecting any pair of points?

R_n = # regions

Last Time

1)



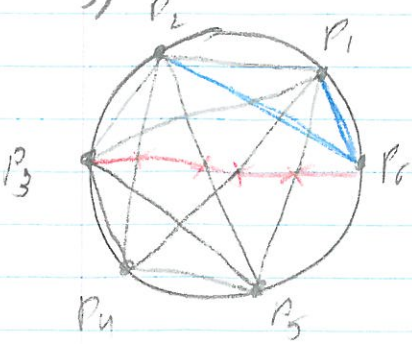
$R_1 = 1$ $R_2 = 2$ $R_3 = 4$ $R_4 = 8$ $R_5 = 16$

- The conjecture $R_n = 2^{n-1}$ is false since $R_6 = 31$

2) • R_n grow polynomially in n, since

$$R_n \le n + \binom{n + \binom{n}{2}}{2} = O(n^4).$$

3)



$$R_n = R_{n-1} + \sum_{k=1}^{n-1} I_k =$$
$$= R_{n-1} + \sum_{k=1}^{n-1} \left[1 + (k-1)(n-k-1)\right]$$
$$= R_{n-1} + (n-1) + \sum_{k=2}^{n-2} (k-1)(n-k-1).$$

$R_1 = 1$

4) Simplifying the Recurrence

$$R_n = R_{n-1} + (n-1) + \sum_{k=2}^{n-2} (k-1)(n-k-1)$$

for $n \geq 2$ and $R_1 = 1$

Claim 4) $\displaystyle\sum_{k=2}^{n-2} (k-1)(n-k-1) = \binom{n-1}{3}$

We show this claim by counting the subsets of $S = \{1, 2, 3, \ldots, n-1\}$ with 3 elements in two different ways.

RHS: There are $\binom{n-1}{3}$ ways to pick three numbers from $S = \{1, 2, \ldots, n-1\}$.

$G_3 = \{$
$\{1,3,4\}$
$\{2,3,4\}$
$\{2,3,5\}$
$\vdots$ $\}$

LHS: • We group the 3-element subsets of S based on their middle element.

• The middle element can take any value $2, 3, \ldots, n-2$.

• For $k = 2, 3, \ldots, n-2$, let G_k be the group whose middle element is k.

• Since the groups are pairwise disjoint

$$\sum_{k=2}^{n-2} |G_k| = \binom{n-1}{3}$$

What is the size of the k-th group G_k?

- Any 3-element subsets in G_k consists of

$$\{\underset{1,2,\dots,k-1}{x}, k, \underset{k+1,\dots,n-1}{y}\}$$

- one element from $\{1,2,\dots,k-1\}$
- the element k
- one element from $\{k+1,\dots,n-1\}$.

By the product rule,

$$|G_k| = \underbrace{|\{1,2,\dots k-1\}|}_{=k-1} \cdot \underbrace{|\{k\}|}_{=1} \cdot \underbrace{|\{k+1,\dots,n-1\}|}_{=(n-k-1)}$$

Therefore,

$$\sum_{k=2}^{n-2}(k-1)(n-k-1) = \sum_{k=2}^{n-2}|G|_k = \binom{n-1}{3} \qquad \square$$

Now our recurrence for the number of region when cutting a circle simplifies to

$$R_n = R_n + (n-1) + \binom{n-1}{3} \text{ if } n \geq 2$$

$$R_1 = 1$$

5) Solving the Recurrence

$$R_n = (n-1) + \binom{n-1}{3} + R_{n-1} \quad \text{for } n \geq 2$$

and $R_1 = 1$.

We apply the unfolding technique:

$$\begin{aligned} R_n &= (n-1) + \binom{n-1}{3} + R_{n-1} \\ &= (n-1) + (n-2) + \binom{n-1}{3} + \binom{n-2}{3} + R_{n-2} \\ &= (n-1) + (n-2) + (n-3) + \binom{n-1}{3} + \binom{n-2}{3} + \binom{n-3}{3} + R_{n-3} \end{aligned}$$

By continuing, we get

$$\begin{aligned} R_n = \; & (n-1) + (n-2) + (n-3) + \dots + 3 + 2 + 1 \\ & + \binom{n-1}{3} + \binom{n-2}{3} + \binom{n-3}{3} + \dots + \binom{3}{3} + \binom{2}{3} + \binom{1}{3} \\ & + R_1 \end{aligned}$$

Since $\binom{2}{3} = \binom{1}{3} = 0$ and $R_1 = 1$,

$$R_n = \left(\sum_{i=1}^{n-1} i\right) + \left(\sum_{k=3}^{n-1} \binom{k}{3}\right) + 1 = 1 + \binom{n}{2} + \sum_{k=3}^{n-1} \binom{k}{3}.$$

Claim 5) $\displaystyle\sum_{k=3}^{n-1} \binom{k}{3} = \binom{n}{4}$.

Claim 5) $\sum_{k=3}^{n-1} \binom{k}{3} = \binom{n}{4}$.

We show this claim by counting the subsets of $S = \{1, 2, 3, \dots, n-1\}$ with 4 elements in a particular way.

RHS: There are $\binom{n}{4}$ ways to pick 4 numbers from $S = \{1, 2, \dots, n\}$

LHS: We group the 4-element subsets of $S = \{1, 2, \dots, n-1\}$ based on their largest element.

- The largest element of four from S can take any value $\underset{3+1}{\overset{}{4}}, \dots, \underset{(n-1)+1}{n}$
- For any k with $3 \leq k \leq n-1$, let G_k contain all 4-element subsets of S whose largest element is $k+1$.

ex $G_3 = \{\{1,2,3,4\}\}$

$G_4 = \{\{1,2,3,5\}, \{1,2,4,5\}, \{1,3,4,5\}, \dots\}$

Since the groups are pairwise disjoint, we have

$$\sum_{k=3}^{n-1} |G_k| = \binom{n}{4}.$$

To determine the size of G_k, we observe that each element of the k-th group has the form.

$$\{\underbrace{a, b, c}_{\text{three elements drawn from } \{1,2,\ldots,k\}}, k+1\}$$

By the product rule $|G_k| = \binom{k}{3} \cdot 1 = \binom{k}{3}$ and,

$$\sum_{k=3}^{n-1} \binom{k}{3} = \sum_{k=3}^{n-1} |G_k| = \binom{n}{4} \qquad \square$$

With claim 5, we arrive at the final form:

$$R_n = 1 + \binom{n}{2} + \sum_{k=3}^{n-1} \binom{k}{3} = 1 + \binom{n}{2} + \binom{n}{4}, \quad n \geq 1.$$

Since $\binom{n}{2} = \Theta(n^2)$ and $\binom{n}{4} = \Theta(n^4)$, we have

$$R_n = \Theta(n^4)$$

In summary, when cutting a circle by drawing all line segments connecting n points on the circle, the circle decomposes into exactly $1 + \binom{n}{2} + \binom{n}{4}$ regions.