5.12

How to choose a random element from a linked list

Given:

A link list L

$\rightarrow \square \rightarrow \square \rightarrow \text{- - - - -} \rightarrow \square \longrightarrow$ nil

- We do not know $|L|$ –
- We are only allowed to make a single pass

Problem: Choose a node of L, uniformly at random. i.e, if $|L| = n$, then each node has $pr = 1/n$ of being chosen.

Algorithm ChooseRandom(L):

```
u:=head(L);
i:=1;
While u ≠ NIL do
    r:= Random(i);        {Returns a random element with
                           uniform probability from {1..i}}
    if r=1 then x:=u;
    u:=succ(u);
    i:=i+1
Return (x).
```

Observe

- In the first iteration $x =$ first element of L with $pr = 1$

- In the second iteration $x =$ second element of L with $pr = 1/2$, and it does not change with ~~$pr = 1/2$~~ $pr = 1/2$

- In the third iteration $x =$ third element of L with $pr = 1/3$; value of x does not change with $pr = 2/3$.

⋮

- In the last iteration $x =$ last element of L with $pr = 1/|L|$; and its value does not change with $pr = |L|-1/|L|$

To Show: Output of chooseRandom(L) is a uniformly random node of List L. Let $|L| = n$, and let v be an arbitrary node of L. We will show, after the algorithm has terminated

$$x = v \quad \text{with} \quad pr = \frac{1}{|L|} = \frac{1}{n}.$$

Pf: Let v be the k^{th} node of L, $1 \le k \le n$.

Note that $x = v$ if and only if

- during the k^{th} iteration, the value of x is set to v.

- for all $i = k+1, k+2, \ldots, n$; the value of x does not change in iteration i.

Define events

$A = \{$"After the algo has terminated $x = v$"$\}$.

For $1 \le i \le n$, define
$A_i = \{$" the value of x changes during the i^{th} iteration"$\}$

Then

$$A = A_k \wedge \overline{A_{k+1}} \wedge \overline{A_{k+2}} \wedge \cdots \wedge \overline{A_n}$$

Since events $A_k, A_{k+1}, \ldots, A_n$ are mutually independent.

$$Pr(A) = Pr\left(A_k \wedge \overline{A_{k+1}} \wedge \overline{A_{k+2}} \wedge \cdots \wedge \overline{A_n}\right)$$

$$= Pr(A_k)\, Pr(\overline{A_{k+1}})\, Pr(\overline{A_{k+2}}) \cdots Pr(\overline{A_n})$$

$$= \frac{1}{k}\left(1 - \frac{1}{k+1}\right)\left(1 - \frac{1}{k+2}\right) \cdots \left(1 - \frac{1}{n}\right)$$

$$= \frac{1}{k} \cdot \frac{k}{k+1} \cdot \frac{k+1}{k+2} \cdot \cdots \cdot \frac{n-1}{n}$$

$$= \frac{1}{n} \quad \blacksquare$$

5.14 Infinite Probability Spaces

Consider the sample space corresponding to coin flips till we obtain a Head. Assume a fair coin is flipped till we obtain a Head.

$S = \{H, TH, TTH, \text{-----}\}$

$= \{T^n H ; n \geq 0\}$

S is infinite set.

Note that the probability that the outcome is $T^n H$ is $\left(\frac{1}{2}\right)^{n+1}$ as we want n tails followed by a Head.

Observe $\displaystyle\sum_{n=0}^{\infty} Pr(T^n H) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} = 1.$

Recall $\displaystyle\sum_{n=0}^{N} x^n = 1 + x + x^2 + \cdots + x^N = \frac{1-x^{N+1}}{1-x}$

(Assuming $x \neq 1$)

If $-1 < x < 1$ then $\displaystyle\lim_{N\to\infty} x^{N+1} = 0$

Thus, $\displaystyle\sum_{n=0}^{\infty} x^n = \lim_{N\to\infty} \sum_{n=0}^{N} x^n$

$\displaystyle= \lim_{N\to\infty} \frac{1-x^{N+1}}{1-x}$

$\displaystyle= \frac{1}{1-x}$

For the coin flips

$$\sum_{n=0}^{\infty} Pr(T^n H) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n.$$

set $x = 1/2$ and then we obtain

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$$

Thus $\sum_{n=0}^{\infty} Pr(T^n H) = \frac{1}{2} \cdot 2 = 1.$

Example 2: Two players; Player 1 & Player 2.

Player 1 & Player 2 alternatively flips a fair coin, and the player who gets the Head first wins. Game starts with Player 1 flipping the coin.

Q: What is the probability that P_1 wins the game?

Note that the sample space

$S = \{T^n H : n \geq 0\}$ as game finishes with the first Head.

Consider the event

$$A = \{\text{"}P_1 \text{ wins the game"}\}$$

$$\Leftrightarrow A = \{T^n H : n \geq 0 \text{ and } n \text{ is even}\}$$

$$= \{T^{2m} H : m \geq 0\}.$$

Thus,

The probability that Player-1 wins the game is.

$$Pr(A) = \sum_{m=0}^{\infty} Pr(T^{2m}H)$$

$$= \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{2m+1}$$

$$= \frac{1}{2} \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{2m}$$

$$= \frac{1}{2} \sum_{m=0}^{\infty} \left(\frac{1}{4}\right)^{m} = \frac{1}{2} \cdot \frac{1}{1-\frac{1}{4}} = \frac{1}{2} \cdot \frac{1}{3/4} = 2/3$$

Let B be the event that Player-2 wins the game.

$$Pr(B) = 1 - Pr(A) = 1 - 2/3 = 1/3 .$$

Let us change the game slightly,

→ The player who flips the second Heads win.

Now $S = \{ T^m H T^n H : m \geq 0, n \geq 0 \}$

The event $A = \{$ "Player-1 wins the game" $\}$

corresponds to

$$A = \{ T^m H T^n H : m \geq 0, n \geq 0, m+n \text{ is odd} \}$$

Split Event A into two events

$A_1 = \{$"Player-1 flips both the first & second heads"$\}$

$A_2 = \{$"Player-2 flips first heads and player-1 flips the second head"$\}$.

$$A_1 = \{T^m H T^n H : m \geq 0, n \geq 0, \text{ m is even and n is odd}\}$$

$$= \{T^{2k} H T^{2l+1} H : k \geq 0, l \geq 0\}$$

$$A_2 = \{T^m H T^n H : m \geq 0, n \geq 0, \text{ m is odd \& n is even}\}$$

$$= \{T^{2k+1} H T^{2l} H : k \geq 0, l \geq 0\}$$

Now $A_1 \cap A_2 = \phi$ and $A_1 \cup A_2 = A$.

$$Pr(A_1) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} Pr(T^{2k} H T^{2l+1} H)$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left(\frac{1}{2}\right)^{2k+2l+3}$$

$$= \frac{1}{2^3} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{2k} \sum_{l=0}^{\infty} \left(\frac{1}{2}\right)^{2l}$$

$$= \frac{1}{8} \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^{k} \sum_{l=0}^{\infty} \left(\frac{1}{4}\right)^{l}$$

$\Rightarrow$

$$Pr(A_1) = \frac{1}{8} \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k \underbrace{\sum_{l=0}^{\infty} \left(\frac{1}{4}\right)^l}_{= \frac{1}{1-\frac{1}{4}} = \frac{4}{3}}$$

$$= \frac{1}{8} \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k \left(\frac{4}{3}\right)$$

$$= \frac{1}{6} \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k = \frac{1}{6} \cdot \frac{4}{3} = 2/9 .$$

Similarly compute

$$Pr(A_2) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} Pr\left(T^{2k+1} H \, T^{2l} H\right)$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left(\frac{1}{2}\right)^{2k+2l+3}$$

$$= 2/9 .$$

Thus $Pr(A) = Pr(A_1) + Pr(A_2) = 4/9$.

and $Pr(B) = Pr(\text{"Player-2 wins"}) = 5/9$.