Random Variables

1. $X : S \to \mathbb{R}$

2. Events & R.V.

Let $X : S \to \mathbb{R}$ be a r.v.
For any real number $x \in \mathbb{R}$, define
"$X = x$" to be the event $\{\omega \in S \mid X(\omega) = x\}$

3. $$\Pr(X=x) = \Pr(\omega \in S \mid X(\omega) = x)$$
$$= \sum_{\omega : X(\omega) = x} \Pr(\omega)$$

4. Independent R.V.

Let $X : S \to \mathbb{R}$ & $Y : S \to \mathbb{R}$ be two r.v.
X & Y are independent if
$\forall x, y \in \mathbb{R}$, the events "$X=x$" and "$Y=y$" are independent, i.e. $\Pr(X=x \wedge Y=y) = \Pr(X=x)\Pr(Y=y)$

5. Expected Value

$$E(X) = \sum_{\omega \in S} X(\omega) \Pr(\omega)$$
$$= \sum_{x} x \cdot \Pr(X=x)$$

RANDOM VARIABLES AND EXPECTATION

Random variable is a function

$X : S \to \mathbb{R}$

Caution

random variable

↓ Not Random | ↓ Not a variable

Example 1: Roll a die twice (Experiment)
observe the sum (function of outcome)

$S = \{(i,j) \mid 1 \le i \le 6,\ 1 \le j \le 6\}$

$X : S \to R$ be the r.v that maps each outcome to sum
i.e., $X(i,j) = i + j$.

Example 2: Flip a fair coin three times and observe the outcome

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

$X : S \to \mathbb{R}$ be the r.v that maps each outcome to number of heads, i.e.,

$X(HHH) = 3$

$X(HTH) = 2$
$X(THH) = 2$
$X(HHT) = 2$

$X(TTH) = 1$
$X(THT) = 1$
$X(HTT) = 1$

$X(TTT) = 0$

Events in context of r.v.

Consider Example 2.

$X=0$ corresponds to the event $\{TTT\} \subseteq S$

$X=1$ corresponds to the event $\{HTT, THT, TTH\} \subseteq S$.

Let S be a sample space, and let $X: S \to \mathbb{R}$ be a random variable.

For any real number x, define "$X=x$" to be the event $\{\omega \in S \mid X(\omega)=x\}$.

Note that

$$Pr(X=0) = Pr(TTT) = 1/8$$

$$Pr(X=1) = Pr(\{THT, HTT, TTH\}) = 3/8$$

$$Pr(X=2) = Pr(\{HHT, THH, HTH\}) = 3/8$$

$$Pr(X=3) = Pr(\{TTT\}) = 1/8$$

$$Pr(X=4) = Pr(\phi) = 0.$$

Thus, $Pr(X=x) = Pr(\{\omega \in S \mid X(\omega)=x\})$

$$= \sum_{\omega : X(\omega)=x} Pr(\omega)$$

What about

$$\Pr(X \geq 2) = \Pr\left(\{\omega \in S \mid X(\omega) \geq 2\}\right)$$

$$= \Pr\left(\{HHT, HTH, THH, HHH\}\right)$$

$$= 4/8 = 1/2.$$

$$= \Pr(X = 2 \vee X = 3)$$

$$= \Pr(X=2) + \Pr(X=3)$$

$$= 3/8 + 1/8 = 1/2.$$

$$\Pr(1 \leq X \leq 2) = \Pr\left(\{\omega \in S \mid 1 \leq X(\omega) \leq 2\}\right)$$

$$= \Pr(X = 1 \text{ or } X = 2)$$

$$= \Pr(X=1) + \Pr(X=2)$$

$$= 3/8 + 3/8 = 6/8 = 3/4.$$

Independent R.V.

Let (S, Pr) be a probability space

Let $X: S \to \mathbb{R}$ and $Y: S \to \mathbb{R}$ to be r.v.

X and Y are independent if for all real numbers x and y, the events $X=x$ and $Y=y$, are independent, i.e.,

$$Pr(X=x \wedge Y=y) = Pr(X=x) \cdot Pr(Y=y)$$

Example: Flip 3 fair coins independently

$X = \#$ heads obtained in the 3 flips.

$$Y = \begin{cases} 1 & \text{if all 3 coins flip to Heads, or all 3 coins flip to Tails} \\ 0 & \text{otherwise} \end{cases}$$

Note that $X: S \to \mathbb{R}$ and $Y: S \to \mathbb{R}$ are random variables.

Note that X and Y are <u>not</u> independent, as $Y=1 \Rightarrow X=0$ or $X=3$.

e.g. $Pr(X=2 \wedge Y=1) = Pr(\phi) = 0$

$$\neq \underbrace{Pr(X=2)}_{= 3/8} \cdot \underbrace{Pr(Y=1)}_{= 1/4}$$

Thus, X and Y are not independent.

Define

$$Z = \begin{cases} 1 & \text{if the first coin flip is Heads} \\ 0 & \text{if the first coin flip is Tails} \end{cases}$$

Claim: Y and Z are independent r.v

Proof: Need to show that for all real numbers y and z

$$Pr(Y=y \wedge Z=z) = Pr(Y=y) \cdot Pr(Z=z)$$

Note that $Pr(Y=1) = 1/4$ $\quad Pr(Z=0) = 1/2$

$Pr(Y=0) = 3/4$ $\quad Pr(Z=1) = 1/2$.

Now,

$$Pr(Y=1 \wedge Z=1) = Pr(\{HHH\})$$
$$= 1/8 = Pr(Y=1) \cdot Pr(Z=1)$$

$$Pr(Y=1 \wedge Z=0) = Pr(TTT)$$
$$= 1/8 = Pr(Y=1) \cdot Pr(Z=0)$$

$$Pr(Y=0 \wedge Z=1) = Pr(\{HHT, HTH, HTT\})$$
$$= 3/8 = Pr(Y=0) \cdot Pr(Z=1)$$

$$Pr(Y=0 \wedge Z=0) = Pr(\{TTH, THT, THH\})$$
$$= 3/8 = Pr(Y=0) \cdot Pr(Z=0)$$

For any value of $y \notin \{0,1\}$ or $z \notin \{0,1\}$,
$Pr(Y=y)=0$ or $Pr(Z=z)=0$, respectively.
Thus r.v Y & Z are independent.

Exercise: Show that r.v. X & Z are not independent.

Generalize the notion of independence to

- Pair-wise independence among n r.v's
- Mutually independent r.v's.

Expected Value

Example 1: $S = \{A, B, C\}$

Probability Function [$Pr(A) = 4/5$, $Pr(B) = 1/10$, $Pr(C) = 1/10$.

Random Variable [$X : S \to \mathbb{R}$ is given by
$X(A) = 1$; $X(B) = 2$; $X(C) = 3$;

What is the "averge value" of X that we will observe, i.e what is the "expected value" of X that we will see when we draw samples from S with the given probabilities.

Intuitively, it should be given by

$$= X(A) \cdot Pr(A) + X(B) \cdot Pr(B) + X(C) \cdot Pr(C)$$

$$= 1 \cdot 4/5 + 2 \cdot 1/10 + 3 \cdot 1/10$$

$$= 13/10$$

(Note that 80% time we observe A, ~~and~~ 10% time we observe B, and 10% time we observe C).

Thus the "averge value" of X is 1.3.

<u>Definition</u>: Let (S, Pr) be a probability space and let $X : S \to \mathbb{R}$ be a r.v. The expected value of X,

$$\boxed{E(X) = \sum_{\omega \in S} X(\omega) \, Pr(\omega)}$$

Example 2: Flipping a fair coin.

Flip a fair coin.

- $S = \{H, T\}$ $\quad Pr(H) = Pr(T) = 1/2$
- Define r.v X as follows

$$X = \begin{cases} 1 & \text{if the coin comes up heads} \\ 0 & \text{if coin comes up tails} \end{cases}$$

Thus $X : S \to \mathbb{R}$ is $X(H) = 1$ and $X(T) = 0$.

$$E(X) = \sum_{\omega \in S} X(\omega) Pr(\omega)$$

$$= X(H)\,Pr(H) + X(T)\,Pr(T)$$

$$= 1 \cdot 1/2 + 0 \cdot 1/2$$

$$= 1/2.$$

[Caution: $E(X)$ is not the value that we expect to observe, as we will never observe $1/2$.
]

Example 3: Rolling a die

Roll a fair die.

$X: \{1,2,3,4,5,6\} \to \{1,2,3,4,5,6\}$

s.t $X(i) = i$.

Then $E[X] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$

$= \frac{7}{2}$.

Define a r.v. $Y: \{1,2,3,4,5,6\} \to \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}\}$

s.t $Y(i) = 1/i$.

Then $E(Y) = 1 \cdot \frac{1}{6} + \frac{1}{2} \cdot \frac{1}{6} + \frac{1}{3} \cdot \frac{1}{6} + \frac{1}{4} \cdot \frac{1}{6} + \frac{1}{5} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{6}$

$= \frac{49}{120}$

(Note: $E(Y) \neq 1/E(X)$, or in general

$E(1/X) \neq 1/E(X)$)

Example 4:

Roll two fair die and define r.v X to be the sum of the outcomes.

i.e $S = \{ (i, j) \mid 1 \le i \le 6,\ 1 \le j \le 6 \}$

outcome of second die

outcome of first die

$X : S \to \mathbb{R}$

such that $X(i,j) = i + j$.

$$E[X] = \sum_{(i,j) \in S} X(i,j) \Pr(i,j)$$

since $\Pr(i,j) = 1/36 \quad \forall\ 1 \le i \le 6,\ 1 \le j \le 6$,

we obtain
$$E[X] = \frac{1}{36} \sum_{(i,j) \in S} X(i,j)$$

$$= \frac{1}{36} \sum_{(i,j) \in S} (i+j)$$

$$= \frac{1}{36} \cdot 252 = 7.$$

An alternate way to compute $E[X]$. Note that

event	# ways	
$X=2$	1	(1,1)
$X=3$	2	(1,2), (2,1)
$X=4$	3	(1,3), (2,2), (3,1)
$X=5$	4	(1,4), (2,3), (3,2), (4,1)
$X=6$	5	..
$X=7$	6	...
$X=8$	5	..
$X=9$	4	..
$X=10$	3	
$X=11$	2	(5,6), (6,5)
$X=12$	1	(6,6)

Therefore

$$E[X] = 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + 4 \cdot \frac{3}{36} + 5 \cdot \frac{4}{36} + 6 \cdot \frac{5}{36} + 7 \cdot \frac{6}{36} + 8 \cdot \frac{5}{36} + 9 \cdot \frac{4}{36} + 10 \cdot \frac{3}{36} + 11 \cdot \frac{2}{36} + 12 \cdot \frac{1}{36}$$

$$= 7$$

This is equivalent to saying that

$$E[X] = \sum_{(i,j)\in S} X(i,j)\, Pr(i,j)$$

$$= \sum_{k=2}^{12} k \cdot Pr(X=k)$$

Lemma 6.4.2

Let (S, Pr) be a probability space and let $X : S \to \mathbb{R}$ be a random variable. The expected value of X is equal to

$$E(X) = \sum_{x} x \cdot Pr(X=x).$$

<u>Proof</u>: Recall that the event "$X=x$" corresponds to $A_x = \{\omega \in S : X(\omega) = x\}$ of the sample space S. Thus,

$$E(X) = \sum_{\omega\in S} X(\omega) \cdot Pr(\omega)$$

$$= \sum_{x} \sum_{\omega : X(\omega)=x} X(\omega) \cdot Pr(\omega)$$

$$= \sum_{x} \sum_{\omega : X(\omega)=x} x \cdot Pr(\omega)$$

$$= \sum_{x} \sum_{\omega \in A_x} x \cdot Pr(\omega)$$

$$= \sum_{x} x \sum_{\omega \in A_x} Pr(\omega)$$

$$= \sum_{x} x \, Pr(A_x)$$

$$= \sum_{x} x \cdot Pr(X = x) \qquad \blacksquare$$

Thus $E(X) = \sum_{x} x \cdot Pr(X = x)$

To evaluate $E(X)$

- determine the values x that X can take
- For each value of x, determine $Pr(X = x)$
- Compute the sum of all products $x \cdot Pr(X=x)$

Example: Suppose a company produces solar panels, where 20% are defective. You buy 3 panels, what is expected value of non-defective panels.

Here sample space $S = \{DDD, DND, NDD, DDN, NND, NDN, DNN, NNN\}$

where N = panel is non-defective | $Pr(N) = 0.8$
D = panel is defective | $Pr(D) = 0.2$

Define $X : S \to \{0, 1, 2, 3\}$

indicating the # of non-defective panels.

e.g $X(DNN) = 2$.

Note that

# non-defective	$Pr(X=x)$
3	$0.8^3 = 0.512$
2	$\binom{3}{2}(0.8)^2(0.2)^1 = 0.384$
1	$\binom{3}{1}(0.8)^1(0.2)^2 = 0.096$
0	$0.2^3 = 0.008$

Thus $E(X) = \sum_x x \cdot Pr(X=x)$

$$= 3 * 0.512 + 2 * 0.384 + 1 * 0.096 + 0 * 0.008$$

$$= 2.4$$

On average 2.4 panels out of 3 are non-defective. This is not a surprise as 80% panels are non-defective.

$$= \underbrace{0.8}_{\text{pr. of non-defective}} * \underbrace{3}_{\text{\# of items}}$$

$= 2.4$ (later on → Binomial Distributions)