1. $X : S \to \mathbb{R}$

2. $$E(X) = \sum_{\omega \in S} X(\omega) \Pr(\omega)$$
$$= \sum_{x} x \cdot \Pr(X=x)$$

3. $$E(X+Y) = E(X) + E(Y)$$

$$E\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i \cdot E(X_i)$$

$$E\left(\sum_{i=1}^{\infty} X_i\right) = \sum_{i=1}^{\infty} E(X_i)$$

(provided some convergence conditions hold)

LINEARITY OF EXPECTATION

4. Geometric Distribution $E[X] = 1/p$

5. Binomial Distribution $E[X] = pn$

6. INDICATOR R.V. $E[X] = \Pr[X=1]$

Back to the 2DIE - sum example

Define $Y : \{1,2,3,4,5,6\} \to \{1,2,3,4,5,6\}$

$Z : \{1,2,3,4,5,6\} \to \{1,2,3,4,5,6\}$.

$$E[Y] = 1 \cdot 1/6 + 2 \cdot 1/6 + 3 \cdot 1/6 + 4 \cdot 1/6 + 5 \cdot 1/6 + 6 \cdot 1/6$$

$$= \frac{7}{2}$$

Similarly

$$E[Z] = \frac{7}{2}.$$

Define $X = Y + Z$

Linearity of expectation.

then $E[X] = E[Y+Z] = E[Y] + E[Z]$

$$= \frac{7}{2} + \frac{7}{2} = 7$$

In general, let $X_1, X_2, \ldots, X_n$ be n r.v. from $X_i : S \to R$

$$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i)$$

"expected value of the sum of random variables is the sum of their expected values"

Geometric Distribution

Experiment: Assume a coin which flips to heads with probability p and flips to tails with probability $1-p$, where $0<p<1$. We flip the coin repeatedly and independently till we obtain Heads.

Sample Space: $S = \{ T^{k-1} H \mid k \geq 1 \}$

Let $X : S \to \mathbb{R}$ be r.v. capturing the total # of flips.

Question: $E[X] = ?$

$$Pr(T^{k-1} H) = \underbrace{p}_{1 \text{ heads}} \underbrace{(1-p)^{k-1}}_{k-1 \text{ tails}}$$

Note that
$$\sum_{k=1}^{\infty} Pr(T^{k-1}H) = \sum_{k=1}^{\infty} p(1-p)^{k-1}$$
$$= p \sum_{k=1}^{\infty} (1-p)^{k-1}$$
$$= p \sum_{l=0}^{\infty} (1-p)^{l}$$
$$= p \cdot \frac{1}{1-(1-p)} = 1.$$

Observe that the event "$X=k$" corresponds to first $(k-1)$ tosses are tails, followed by heads.

Thus, $Pr(X=k) = Pr(T^{k-1}H) = p(1-p)^{k-1}$.

Therefore,

$$E(X) = \sum_{k=1}^{\infty} k \cdot \Pr(X=k)$$

$$= \sum_{k=1}^{\infty} k p (1-p)^{k-1}$$

$$\overset{?}{=} 1/p \quad \blacksquare$$

How to show this?

Consider $\displaystyle\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ for $-1 < x < 1$.

Differentiate both sides:

$$\sum_{k=0}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2}$$

$$\Leftrightarrow \quad \sum_{k=1}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2} \qquad \left(\text{as } k=0, \quad 0 \cdot x^{-1} \equiv 0\right)$$

Set $x = 1-p$ and we obtain

$$\sum_{k=1}^{\infty} k (1-p)^{k-1} = \frac{1}{\left(1-(1-p)\right)^2} = \frac{1}{p^2}.$$

Thus $\displaystyle E(X) = \sum_{k=1}^{\infty} k p (1-p)^{k-1} = p \sum_{k=1}^{\infty} k (1-p)^{k-1}$

$$= p \cdot \frac{1}{p^2} = \frac{1}{p}.$$

Theorem (Geometric Distribution)

Let p be a real number, $0 < p < 1$.

A r.v X has a geometric distribution with parameter p,

if $Pr(X = k) = p(1-p)^{k-1}$ for $k \geq 1$.

Then $E(X) = 1/p$.

Theorem (Binomial Distribution)

Let $n \geq 0$, $0 < p < 1$. A random variable X has a binomial distribution with parameters n and p, if

$$Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

and

$$E[X] = pn.$$

BINOMIAL DISTRIBUTION

Experiment: Flip a coin, independently n-times.
Let $Pr(H)=p$, $Pr(T)=1-p$.

Define a r.v. $X : S \to \mathbb{R}$
such that it corresponds to # times the toss comes up heads.

Note that X takes values from $\{0,1,2,3,\ldots,n\}$.

$$E(X) = \sum_{k=0}^{n} k \cdot Pr(X=k)$$

$Pr(X=k)$ = What is the probability that in n-tosses we have exactly k-heads?

$$= \binom{n}{k} p^k (1-p)^{n-k}$$

We are looking at sequences of length n containing exactly k heads.
# of such sequences are $\binom{n}{k}$.
Each of them occurs with $pr = p^k (1-p)^{n-k}$

Check: $$\sum_{k=0}^{n} Pr(X=k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$
$$= ((1-p)+p)^n$$
$$= 1$$

from Newton-Binomial Thm

Now

$$E(X) = \sum_{k=0}^{n} k \cdot P_r(X=k)$$

$$= \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

Note that $k \cdot \binom{n}{k} = k \cdot \frac{n!}{k!(n-k)!} = \frac{\cancel{k} \cdot n \cdot (n-1)!}{\cancel{k} \cdot (k-1)!(n-k)!}$

$$= n \cdot \binom{n-1}{k-1}$$

$$= \sum_{k=1}^{n} n \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

set $l = k-1$, we obtain

$$= \sum_{l=0}^{n-1} n \binom{n-1}{l} p^{l+1} (1-p)^{n-1-l}$$

$$= np \sum_{l=0}^{n-1} \binom{n-1}{l} p^l (1-p)^{n-1-l}$$

$$= np \left[p + (1-p) \right]^{n-1}$$

Newton - Binomial Thm

$$E[X] = np .$$

An aternative approach.

Define r.v. $X_1, X_2, \cdots, X_n$ such that

$$X_i = \begin{cases} 1 & \text{if } i\text{-th coin flip is Heads} \\ 0 & \text{if } i\text{-th coin flip is Tails} \end{cases}$$

Note that $X = X_1 + X_2 + \cdots + X_n$.

Thus $E[X] = E[X_1 + X_2 + \cdots + X_n]$

$$= E[X_1] + E[X_2] + \cdots + E[X_n].$$

We have already seen, for $1 \leq i \leq n$

$$E[X_i] = 1 \cdot Pr[X_i = 1] + 0 \cdot Pr[X_i = 0]$$
$$= 1 \cdot p = p$$

Thus $E[X] = \underbrace{p + p + \cdots + p}_{n\text{-times}} = pn$

INDICATOR R.V.

A r.v. X is an indicator r.v, if X only takes values in $\{0,1\}$.

Claim: If X is an indicator r.v, then

$$E[X] = Pr[X=1]$$

Pf: By definition

$$E[X] = \sum_k k \cdot Pr(X=k)$$
$$= 0 \cdot Pr(X=0) + 1 \cdot Pr(X=1)$$
$$= Pr(X=1). \quad \square$$

Example: Runs in a Random Bitstrings

Let $S = s_1 s_2 \cdots s_n$ be a random bitstring of length n, such that $s_i \in \{0,1\}$, $1 \le i \le n$.

Let $k \ge 1$ be an integer.

Run of length k in S is a consecutive sequence of k-bits, all of them are either 0, or all of them are 1.

Define r.v X = # runs of length k in S.

 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21

Example 1: $S = 110010011110111000011$

$k = 3$

Then we have runs of length 3 at positions starting at 8, 9, 13, 16, and 17.

Thus $X(s) = 5$

Question: $E(X) = ?$

<u>Observe</u>: Run of length k can start at any position $1, 2, 3, \ldots, n-k+1$.

Define $X_i = \begin{cases} 1 & \text{if there is a run of length } k \text{ starting at position } i \\ 0 & \text{otherwise.} \end{cases}$

$$X = X_1 + X_2 + \cdots + X_{n-k+1}.$$

$$E(X_i) = Pr(X_i = 1) = \underbrace{\left(\tfrac{1}{2}\right)^k}_{k\ 0\text{'s}} + \underbrace{\left(\tfrac{1}{2}\right)^k}_{k\ 1\text{'s}} = \frac{1}{2^{k-1}}$$

$$E(X) = E(X_1 + X_2 + \cdots + X_{n-k+1})$$

Linearity of expectation

$$= E(X_1) + E(X_2) + \cdots + E(X_{n-k+1})$$

$$E(X) = \frac{n-k+1}{2^{k-1}}$$

Comments

- Suppose $k = 1 + \log n$.

  then $$E(x) = \frac{n-k+1}{2^{k-1}}$$

  $$= \frac{n - 1 - \log n + 1}{2^{1+\log n - 1}}$$

  $$= \frac{n - \log n}{n} = 1 - \frac{\log n}{n}.$$

  Thus, for large values of n, expected # runs of of length $1+\log n$ is close to 1.

- Suppose $k = 1 + \frac{1}{2}\log n$.

  then $$E(x) = \frac{n-k+1}{2^{k-1}}$$

  $$= \frac{n - \frac{1}{2}\log n}{2^{\frac{1}{2}\log n}} = \sqrt{n} - \frac{\log n}{2\sqrt{n}}.$$

  Thus, there are about $\sqrt{n}$ runs of of length $1 + \frac{1}{2}\log n$ for large values of n.