# Probabilistic Method

- MAX CUT
- RAMSEY NUMBERS $R(K,K)$
- HAMILTONIAN PATHS IN TOURNAMENTS
- Sperner's Thm

# Probabilistic Method

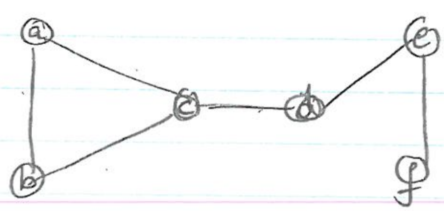
Let $G = (V, E)$ be a simple graph.

vertices  edges.

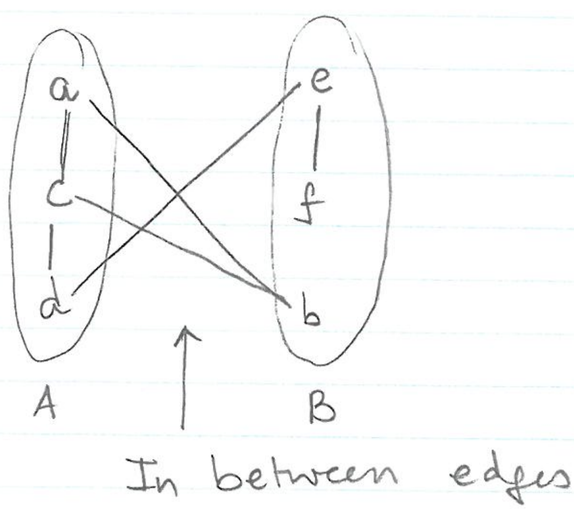
Let $V = A \cup B$ and $A \cap B = \phi$.

We say an edge $e \in E$ is between $A$ and $B$ if one of its endpoint is in $A$ and other is in $B$.

Consider the following graph




Define $A$ & $B$ such that




Notice that in this case at least half of the edges of $G$ are "between" edges.

Claim: Let $G=(V,E)$ be a simple graph with $m$ edges. The vertex set $V$ can be partitioned into two subsets $A$ and $B$ such that the number of edges between $A$ and $B$ is at least $m/2$.

Proof: Take a fair coin and toss it $|V|$ times, once for each vertex. If the outcome is Heads, add the vertex $v \in V$ to $A$, otherwise add it to $B$.

Define r.v. $X$ = # edges between $A$ & $B$.

Let $E = (e_1, e_2, \dots, e_m)$ be the edges of $G$.

Define $m$ 0-1 r.v $X_1, X_2, \dots, X_m$ as follows

$$X_i = \begin{cases} 1 & \text{if } e_i \text{ is an edge between } A \text{ \& } B \\ 0 & \text{otherwise.} \end{cases}$$

$$X = \sum_{i=1}^{m} X_i$$

$$E(X) = E\left(\sum_{i=1}^{m} X_i\right) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{m} \Pr(X_i=1)$$

G(n,p) properties:

1. $np < 1$ No connected components of size larger than $O(\log n)$
2. $np = 1$ largest conn comp of size $n^{2/3}$
3. $np = c > 1$ unique giant component with a fraction of vertices – no other components have more than $O(\log n)$ vertices
4. $p < \frac{(1-\epsilon)\ln n}{n}$ $G(n,p)$ will have isolated vertices
5. $p > \frac{(1+\epsilon)\ln n}{n}$ $G(n,p)$ will be connected.

Let the endpoints of $e_i = (u, v)$ where $u, v \in V$.

| | | |
|---|---|---|
| Case 1: | $u \in A, v \in A$ | $X_i = 0$ |
| Case 2: | $u \in A, v \in B$ | $X_i = 1$ |
| Case 3: | $u \in B, v \in A$ | $X_i = 1$ |
| Case 4: | $u \in B, v \in B$ | $X_i = 0$ |

Since $X_i = 1$ in two out of four cases, and each case is equally likely to occur, thus we have

$$E(X_i) = Pr(X_i = 1) = 1/2.$$

$$\Rightarrow E(X) = Pr(X_1 = 1) + Pr(X_2 = 1) + \cdots + Pr(X_m = 1)$$

$$= m/2. \quad — \quad *{*}*$$

Suppose that the claim in theorem is false.

$\Rightarrow$ No matter how we partition the vertices in sets $A$ & $B$, $A \cap B = \phi$ and $A \cup B = V$, the # inbetween edges $< \frac{m}{2}$.

$\Rightarrow E(X) < \frac{m}{2}$ and that contradicts $*{*}*$ ∎

/ Comment: 2 approx to MAX-CUT /

Example 2:

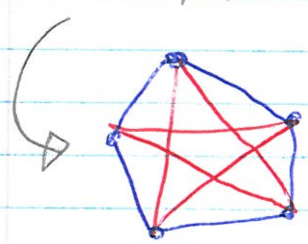
RAMSEY THEORY.

Recall that $R(k,l)$ for integers $k\geq 0$, $l\geq 0$ is defined to be

$$R(k,l) = \min\{n \mid \text{any graph on } n \text{ vertices whose edges are colored red or blue always contains either a red clique of size } k \text{ or a blue clique of size } l\}.$$

In the very first class we saw that

$R(3,3) > 5$ and $R(3,3) = 6$



No red or blue triangle

Claim: For any $k \geq 3$, $R(k,k) > \lfloor 2^{k/2} \rfloor$

Proof: Consider a complete graph on $n$ vertices, and color each edge red or blue with $pr = 1/2$, independent of any other edge.
(e.g. flip a fair coin for each edge
"heads" $\rightarrow$ color it red
"tails" $\rightarrow$ color it blue)

Let $G=(V,E)$ be the complete bichromatic graph.

Take a set $S \subseteq V$ of $k$ vertices.

$\Pr(\text{all edges in the graph induced by } S \text{ are red}) = \left(\frac{1}{2}\right)^{\binom{k}{2}}$.

$\Pr(\text{all edges in the graph induced by } S \text{ are blue}) = \left(\frac{1}{2}\right)^{\binom{k}{2}}$

In all there are $\binom{n}{k}$ possible subsets of $k$ vertices out of $n$ vertices.

Thus

$\Pr(G \text{ contains a red clique or a blue clique of size } k) \leq 2 \cdot \left(\frac{1}{2}\right)^{\binom{k}{2}} \binom{n}{k}$

This follows from union bound

~~What value of $n$ ensures that~~

~~$\Pr(\quad) < 1$ ?~~

Note that $\binom{n}{k} \leq n^k$

If $2\binom{n}{k} \leq 2n^k < 2^{k(k-1)/2}$ then $\Pr(\quad) < 1$.

i.e if $n \leq 2^{k/2 - 1}$ then

$2 \cdot \left(2^{k/2-1}\right)^k = 2\left(2\right.$

Consider the quantity

$$2 \cdot \left(\frac{1}{2}\right)^{\binom{k}{2}} \binom{n}{k}$$

$$= \frac{2\binom{n}{k}}{2^{\binom{k}{2}}} \quad — \; (1)$$

We are interested to know what value of $n$ will ensure that (1) is $< 1$.

$\Rightarrow$ Pr ( G does not contain a red clique of size k and G does not contain a blue clique of size k ) $> 0$.

Set $n := \lfloor 2^{k/2} \rfloor \leq 2^{k/2}$

Then (1) becomes

$$\frac{2\binom{n}{k}}{2^{\binom{k}{2}}} = \frac{n(n-1)(n-2) \cdots (n-k+1)}{k!} \cdot \frac{2}{2^{(k^2-k)/2}}$$

$$\leq \frac{n^k}{k!} \cdot \frac{2^{1+k/2}}{2^{k^2/2}}$$

$$\leq \frac{\cancel{(2^{k/2})^k}}{k!} \cdot \frac{2^{1+k/2}}{\cancel{2^{k^2/2}}} = \frac{2^{1+k/2}}{k!}$$

but $k! > 2^{1+k/2}$ for $k \geq 3$. (Pf. by induction on k)

Thus $$\frac{2\binom{n}{k}}{2^{\binom{k}{2}}} < 1$$

and thus $R(k,k) > 2^{k/2 - 1}$ ∎

Implications:

Let $k = 20$ $n = 1024$

Let n represents people,

red edge – the corresponding individuals are mutual friends
blue edge – ———— ———— strangers.

Theorem $R(k,k) > \lfloor 2^{k/2} \rfloor = \lfloor 2^{10} \rfloor = 2^{10} = 1024$

$\Rightarrow \exists$ a group of 1024 people that does not contain a subgroup of 20 mutual friends or 20 strangers.

Infact even if there is a pr $= 1/2$ that any two individuals are friends or strangers, the probability that there is a subgroup of 20 mutual friends or 20 mutual strangers has probability $\leq \frac{2^{1+k/2}}{k!} = \frac{2^{11}}{20!} = 0.\underbrace{000\cdots0}_{15\text{ zeros}}84\ldots$

## Example 3: Hamiltonian Paths.

A Hamiltonian Path in a directed graph $G$ is a directed path that passes through all vertices of $G$.

<u>Claim</u>: Each edge of a complete graph $G$ on $n$-vertices can be directed in such a way that $G$ has at least $\frac{n!}{2^{n-1}}$ directed Hamiltonian paths.

<u>Pf</u>: Consider a complete graph on $G=(V,E)$ $n$-vertices.

For each edge $e=(uv)\in E$, $u\in V$, $v\in V$, toss a fair coin (independently of other edges).

<u>If the coin comes up</u> heads orient $e$ from $u$ to $v$.
tails orient $e$ from $v$ to $u$.

Consider a permutation of $n$ vertices — and let that permutation be
$\alpha = (\alpha(1), \alpha(2), \dots, \alpha(n))$.

If in $G$, each edge $\alpha(i)\alpha(i+1)$ for $1\le i\le n-1$, appears in that orientation, then

$(\alpha(1), \alpha(2), \dots, \alpha(n))$ is a Hamiltonian path.

Let $X_\alpha = \begin{cases} 1 & \alpha \text{ is a Hamiltonian path.} \\ 0 & \text{otherwise} \end{cases}$

$$E[X_\sigma] = \Pr\left(\text{Each edge } \sigma(i)\sigma(i+1) \text{ is oriented in this way for } \forall 1 \le i \le n-1\right)$$

$$= \frac{1}{2^{n-1}}$$ as each edge orientation is chosen independently.

Thus total # of Hamiltonian paths $X$ equals to

$$E[X] = E\Big[\sum_{\sigma \in \text{permutation of } n} X_\sigma\Big] = \sum_\sigma E[X_\sigma]$$

$$= \frac{n!}{2^{n-1}}$$

If expected # of directed Hamiltonian paths in a "randomly-oriented graph" is $\frac{n!}{2^{n-1}}$, then there is always a way to orient the graph such that # directed Hamiltonian paths $\geq \frac{n!}{2^{n-1}}$.

# Sperner's Theorem

Let $n \geq 1$ be an integer and let $S = \{1, 2, 3, \dots, n\}$.
Let $S_1, S_2, \dots, S_m$ be a sequence of $m$ subsets of $S$ such that

$$\forall i, j,\ i \neq j,\quad S_i \not\subseteq S_j \text{ and } S_j \not\subseteq S_i.$$

Then $$m \leq \binom{n}{\lfloor n/2 \rfloor}. \quad \text{— } (1)$$

Proof: Assume none of the subsets $S_1, S_2, \dots, S_m$ is empty, otherwise $m = 1$ and (1) holds trivially.

Choose a random permutation of elements of $S$.

e.g. for $n = 4$, let a permutation be $\{3, 1, 4, 2\}$.

Define sets
$$\left.\begin{aligned} A_1 &= \{3\} \\ A_2 &= \{1, 3\} \\ A_3 &= \{1, 3, 4\} \\ A_4 &= \{1, 2, 3, 4\} \end{aligned}\right\} \text{Random subsets of } S \text{ as they are based on a random permutation}$$

If any of $\{S_1, S_2, \dots, S_m\}$ equals any of $\{A_1, A_2, \dots, A_n\}$ we say "some $S_i$ occurs in the sequence $A_1, A_2, \dots, A_n$".

Clearly, at most one $S_i$ can occur in $A_1, A_2, \dots, A_n$ as $A_1 \subseteq A_2 \subseteq A_3 \dots \subseteq A_n$.

Define r.v. $X = \begin{cases} 1 & \text{if any of } S_i \text{ occurs in } A_1, A_2, \dots, A_n \\ 0 & \text{otherwise} \end{cases}$

$E(X) \le 1.$

For each $1 \le i \le m$, define

$$X_i = \begin{cases} 1 & \text{if } S_i \text{ occurs in sequence } A_1, A_2, \ldots, A_n \\ 0 & \text{otherwise.} \end{cases}$$

Let $k = |S_i|$

Note that $X_i = 1$ iff $S_i = A_i$.

Let $A_k = \{a_1, a_2, \ldots, a_k\} \subseteq S$.

Hence, $X_i = 1$ iff the first $k$ values in the random permutation form a permutation of the elements of $S_i$

$$\boxed{\underbrace{a_1, a_2, \ldots a_k}_{\text{permutation of } S_i} \,\Big|\, a_{k+1} \cdots\cdots a_n}$$

In all there are $k!\,(n-k)!$ total # of permutations of $n$-elements, where first $k$ elements correspond to the set $S_i$.

Thus, $E(X_i) = \Pr(X_i = 1)$

$$= \frac{k!\,(n-k)!}{n!} = \frac{1}{\binom{n}{k}} = \frac{1}{\binom{n}{|S_i|}}$$

Now

$$X = \sum_{i=1}^{m} X_i$$

$$E(X) = E\left(\sum_{i=1}^{m} X_i\right)$$

$$= \sum_{i=1}^{m} E(X_i)$$

$$= \sum_{i=1}^{m} \frac{1}{\binom{n}{|S_i|}}$$

Also, we know $E(X) \le 1$.

Thus

$$\sum_{i=1}^{m} \frac{1}{\binom{n}{|S_i|}} \le 1$$

Note that $\binom{n}{|S_i|} \le \binom{n}{\lfloor \frac{n}{2} \rfloor}$ as largest value in a binomial is in the middle (see Pascals $\triangle$)

Hence $1 \ge \displaystyle\sum_{i=1}^{m} \frac{1}{\binom{n}{|S_i|}} \ge \sum_{i=1}^{m} \frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} = \frac{m}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$

or $\binom{n}{\lfloor \frac{n}{2} \rfloor} \ge m$ ∎