Assignment 4

#2. Geometric distribution with parameter $1/2300$

Expected # integers that need to be selected $= \dfrac{1}{1/2300}$

~ 2300

#3 $X =$ Number of games played

Note that $X \in \{3, 4, 5\}$

$$Pr(X=3) = \underbrace{\left(\tfrac{1}{4}\right)^3}_{\text{A wins first 3}} + \underbrace{\left(\tfrac{3}{4}\right)^3}_{\text{B wins first 3}} \cong 0.43$$

disjoint events

$Pr(X=4) =$ Either A wins the 4^{th} game and game ends

or

B wins the 4^{th} game and game ends

If A wins the 4^{th} game, then A must have won 2 out first 3 games

$$Pr \rightarrow \underbrace{3\left(\tfrac{1}{4}\right)^2\left(\tfrac{3}{4}\right)}_{\text{A winning 2 out of first 3 games}}\left(\tfrac{1}{4}\right) \leftarrow \text{A winning last game}$$

Similarly, if B wins the 4^{th} game

$$Pr \rightarrow 3\left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right)^2\left(\tfrac{3}{4}\right)$$

Therefore,

$$Pr(X=4) = 3\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right) + 3\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)$$

$$\simeq 0.35$$

What is $Pr(X=5)$

$$\hookrightarrow 1 - \left[Pr(X=3) + Pr(X=4)\right] \simeq 0.21$$

or

$$\underbrace{\binom{4}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^2}_{\text{A wins 2 games out of first 4.}}\underset{\uparrow\ \text{A wins 5th game}}{\left(\frac{1}{4}\right)} + \binom{4}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^2\underset{\uparrow\ \text{B wins 5th game}}{\left(\frac{3}{4}\right)}$$

Thus

$$E[X] = 3 \cdot Pr(X=3) + 4\,Pr(X=4) + 5\,Pr(X=5)$$

$$\sim 3.77 \text{ games.}$$

# of boxes bought till first car is obtained (=1)

#4: ⓐ $X = \sum_{i=0}^{n-1} X_j = X_0 + X_1 + \cdots\cdots + X_{n-1}$

# of boxes bought till a car different than first one is obtained.

# of boxes need to be bought till we obtain all types of cars

additional boxes bought to obtain 2nd car

additional boxes bought to obtain 3rd car.

clearly $X = X_0 + X_1 + X_2 + \cdots + X_{n-1}$

boxes bought to obtain first car

ⓑ Once j distinct types have been obtained, there are $n-j$ types still remaining out of possible n-types.

Thus the pr of success on the next purchase (i.e obtaining a new type of car) $= \dfrac{n-j}{n}$.

as boxes are equally likely to contain any of the possible cars.

ⓒ X_j follows geometric distribution with parameter $\dfrac{n-j}{n}$

ⓓ $E(X_j) = \dfrac{n}{n-j}$

Linearity of expectation

ⓔ Thus $E(X) = E\left(\sum_{j=0}^{n-1} X_j\right) = \sum_{j=0}^{n-1} E(X_j)$

Thus

$$E(X) = \sum_{j=0}^{n-1} \frac{n}{n-j}$$

$$= n\left(\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \cdots + \frac{1}{1}\right) = n\sum_{i=1}^{n} \frac{1}{i}$$

$$\sim n H_n$$

(e) $\sum_{i=1}^{n} \frac{1}{j} \approx \ln n + 0.5772$

Thus when $n = 100$, $\ln n + 0.5772 \approx 518$ boxes

5. (a) $E(XY) = E(X)E(Y)$ if X & Y are indep. r.v.

Pf: $$E(XY) = \sum_{k \in XY} k \cdot Pr(XY = k)$$

by Definition of $E(\ )$

$$= \sum_{k_1 \in X_1,\ k_2 \in Y} k_1 k_2 \cdot Pr(X = k_1 \wedge Y = k_2)$$

Independence of X & Y

$$= \sum_{k_1 \in X} \sum_{k_2 \in Y} k_1 k_2\, Pr(X = k_1) \cdot Pr(Y = k_2)$$

$$= \sum_{k_1 \in X} k_1\, Pr(X = k_1) \sum_{k_2 \in Y} k_2 \cdot Pr(Y = k_2)$$

$$= \sum_{k_1 \in X} k_1\, Pr(X = k_1)\, E(Y)$$

$$= E(Y) \sum_{k_1 \in X} k_1 \Pr(X = k_1)$$

$$= E(Y) \cdot E(X) \quad \blacksquare$$

(b) $X =$ # heads in 2 flips

$Y =$ # tails in 2 flips

$$E(X) = 2 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{4} = 1$$

Similarly $E(Y) = 1$.

$XY = 1$ only when we have one tails and one heads, otherwise it is 0.

$$E[XY] = 1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = 1/2$$

Thus $E[XY] = 1/2 \neq E[X]E[Y] = 1$ $\blacksquare$

#6. Bonus problem.

$F(x) - G(x) = 0$ can have at most d roots.

If a random chosen number in $[1 .. 100d]$ turns out to be root of $F(x) - G(x)$, then we cannot conclusively say that $F(x) = G(x)$.

$$\Pr(\text{Selecting a root of } F(x) - G(x) \text{ in } 1..100d) \leq \frac{d}{100d}$$

To increase the pr. of success,

increase the range — say [1.. 10000 d].

then pr of error $< 10^{-4}$ ■

1.1c 20 integers → 5 even ; 15 odd

7 element subsets with 3 even integers

Product rule:

Task 1: Choose 3 even integers

$\# \text{ways} = \binom{5}{3}$

Task 2: choose 4 odd integers

$\# \text{ways} = \binom{15}{4}$

$\# \text{ ways to execute procedure} = \binom{5}{3} \cdot \binom{15}{4}$

Note: Order doesn't matter as we are looking at subsets of 7 elements.

(and not strings of 7 elements).

1.2 (a) $\underbrace{\overset{5\text{ even}}{\binom{5}{5}}\overset{2\text{ odd}}{\binom{15}{2}} + \overset{2\text{ even}}{\binom{5}{2}}\overset{5\text{ odd}}{\binom{15}{5}} + \overset{1\text{ even}}{\binom{5}{1}}\overset{6\text{ odd}}{\binom{15}{6}} + \binom{5}{0}\overset{7\text{ odd}}{\binom{15}{7}}}_{\text{disjoint cases}}$

1.3 b) $2 \cdot 2^{73} - 2^{69}$

Inclusion

A = # of bit strings of length 77 that start with 1111

B = # of bit strings of length 77 that end with 0000.

$A \cap B$ = # of bit strings of length 77 that start with 1111 & end with 0000.

thus

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$= 2^{73} + 2^{73} - 2^{69}$$
$$= 2^{74} - 2^{69}$$

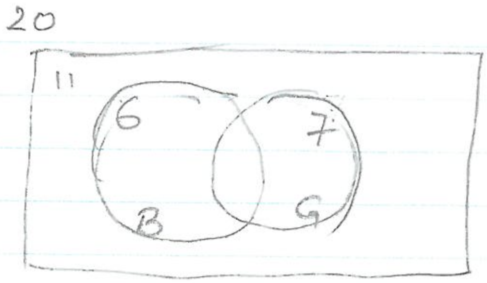
1.4 b) $(2x - 7y)^{15}$

$$= (2x + (-7y))^{15}$$

Term with $x^4y^{11} \rightarrow \binom{15}{11}(2x)^4(-7y)^{11} = \binom{15}{11}2^4(-7)^{11}$

$$= -\binom{15}{11}2^4 7^{11}.$$

1.5 b)



$$20 = 11 + |B| + |G| - |B \cap G|$$

$$\Leftrightarrow |B \cap G| = 11 + 6 + 7 - 20 = 24 - 20 = 4$$

1.6 d) Self-explanatory

1.7 b) $2S_{n-1} + S_{n-2}$ → add 2 a's to strings of length n-2 that contain odd # of a's.

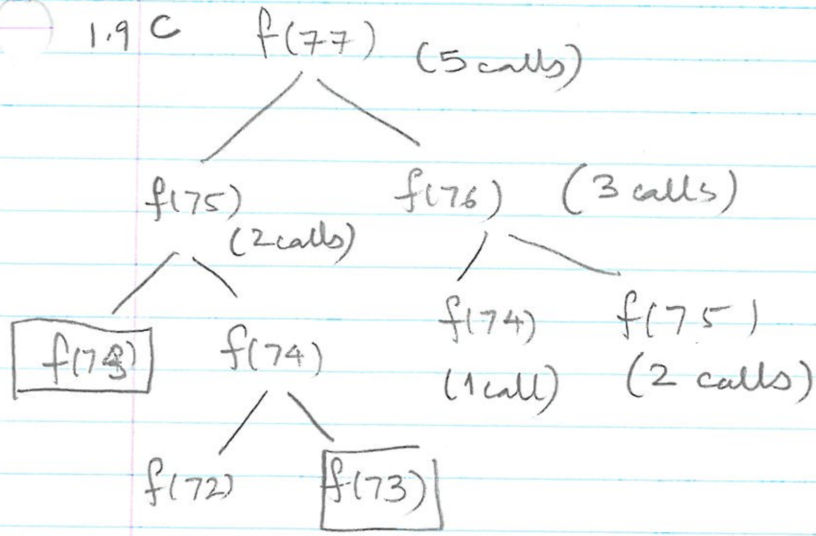
add b or c to strings of length n-1 that contain odd # a's

correct answer → d) $\underbrace{2S_{n-1}}_{} + (\underbrace{3^{n-1} - S_{n-1}}_{\text{\# strings of length n-1 containing even \# of a's}})$

\# strings of length n-1 containing even # of a's.

1.8 b	$2n^2+7$	$3n^2+7$	$4n^2+7$	None of the above
$f(0)=7$	7 ✓	7 ✓	7 ✓	
$f(1)=10$	9 X	10 ✓	11 X	
$f(2)=19$		19 ✓		
$f(3)=34$		34 ✓		

1.9 C



$\Rightarrow$ f(74) needs 1 call to f(73)

$\Rightarrow$ f(75) needs 2 calls to f(73)

C: Total of 5 calls.

1.10 C: Note that (1, 6) is divisible by 7.

any time we either add 7 by either $\begin{cases} 3+4 \\ 5+2 \end{cases}$

hence ~~a+~~ (a, b) is div by 7.

Moreover $a>0$, $b>0$.

1.11 $\binom{102}{3}$ $99+3=102$ includes 99 0's and 3 1's

1.12 $Pr(B) = 2/3$

$Pr(A|B) = 4/5$

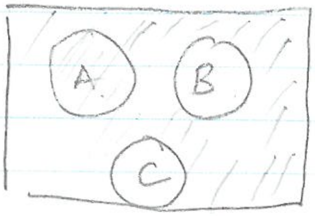
$Pr(A \cap B) = ?$

Baye's Thm:

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$$

or $Pr(A \cap B) = Pr(A|B)\,Pr(B) = \frac{4}{5} \cdot \frac{2}{3} = \frac{8}{15}$

1.13 False - Make a diagram



1.14 $1 - \binom{70}{20} \Big/ \binom{100}{20} \rightarrow$ Pr of choosing all negatives.

1.15 (a)

$$\frac{1}{\binom{100}{20}} \sum_{k=0}^{10} \binom{70}{2k} \binom{30}{20-2k}$$

Captures disjoint sets.

choose even # of negative numbers.

choose remaing +ve numbers.

~~Simple~~

1.16 $Pr(0) = 1/3$

$Pr(1) = 2/3$

15 0's in a bit string of length 77.

$$= \binom{77}{15} \left(\frac{1}{3}\right)^{15} \left(\frac{2}{3}\right)^{62}$$

1.17. $Pr(A|B)$ length 5 bit strings

A - Exactly two 0's

B - Even # of 0's

How many ways we can form even # of 0's

# zeros	0 - zeros	2 - zeros	4 - zeros
# ways	1	$\binom{5}{2}$	$\binom{5}{4}$

Hence we are looking at Pr(2-zeros) out of possible

choices $\equiv \dfrac{\binom{5}{2}}{1 + \binom{5}{2} + \binom{5}{4}} = \dfrac{\binom{5}{2}}{2^4}$

1.18 b

1.19 ~~E(X) =~~

$$x_i = \begin{cases} 3 & \text{if toss is heads} \\ -1 & \text{if toss is tails} \end{cases}$$

$$X = x_1 + x_2 + \cdots + x_n$$

$$E(X) = E(x_1 + x_2 + \cdots + x_n)$$

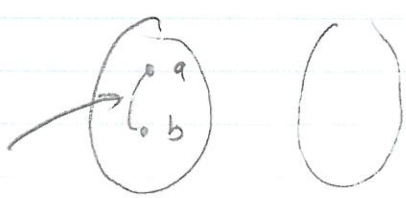
$$= E(X_1) + E(X_2) + \cdots + E(X_n)$$

$$= \left[3 \cdot Pr(H) + -1\, Pr(T)\right] * n$$

$$= \left(3 \cdot \tfrac{1}{2} - \tfrac{1}{2}\right) n = n$$

$$E(X_i) = 3 \cdot \tfrac{1}{2} - 1 \cdot \tfrac{1}{2} = 1$$

1.20 m/4



V' $\quad$ $V - V'$

What is $pr\left[e = (ab) \text{ has both of its end points in } V'\right]$

$$= \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{4}.$$

Define $X_{ab} = \begin{cases} 1 & \text{if } e_{ab} \in V' \\ 0 & \text{otherwise} \end{cases}$

then

$$X = \sum_{(ab)\in E} X_{ab}$$

$$E(X) = \sum_{(ab)\in E} E(X_{ab})$$

$$= m \cdot \frac{1}{4}$$

1.21 : X & Y are independent
if for all real numbers x & y

the events $X=x$ and $Y=y$ are independent.
i.e $Pr(X=x \wedge Y=y) = Pr(X=x) \cdot Pr(Y=y)$
~~X = # of 0s~~

X = # of 0s in bit strings of length 5

Y = value of the first bit.

Consider $x=5$, $y=1$.

Now $Pr(X=x) = \left(\frac{1}{2}\right)^5$

$Pr(Y=y) = 1/2$

$Pr(X=x \wedge Y=y) = 0$ as if we have 5 0's, first bit can't be 1.

Thus X & Y are not independent since

$Pr(X=x \wedge Y=y) \neq Pr(X=x) \cdot Pr(Y=y)$

1.22 $E(L) = 100 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = 50$

$\max(L,S) = 100$

So $E(\max(L,S)) = 100$

1.23. Two tails (k)

$k=1$

returns 2 with $pr = 1/4$

returns 2^2 with $pr = \frac{3}{4} \cdot \frac{1}{4}$

2^3 with $pr = \left(\frac{3}{4}\right)^2 \cdot \frac{1}{4}$

$\vdots$

2^k with $pr = \left(\frac{3}{4}\right)^{k-1} \cdot \frac{1}{4}$

geom distributi

1.24. $pr(\text{Sum}=12) = 1/36$

geometric distribution with parameter $= 1/36$.

$E(X) = 1/\tfrac{1}{36} = 36$.