BIJECTION RULE: If A and B are finite sets, and $\exists$ a bijection $f : A \to B$, $|A| = |B|$.

Problem: How many subsets are there for a set consisting of n-elements?

COMPLEMENT RULE: If U is finite and $A \subseteq U$, then $|A| = |U| - |U \setminus A|$.

Problem: Valid passwords in UNIX

SUM-RULE: If $A_1, A_2, \ldots, A_m$ are finite and pairwise disjoint sets then

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$

Problem: Valid passwords in UNIX

INCLUSION-EXCLUSION

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

PERMUTATIONS

A permutation is an ordered sequence of elements of set, where each element occurs once.

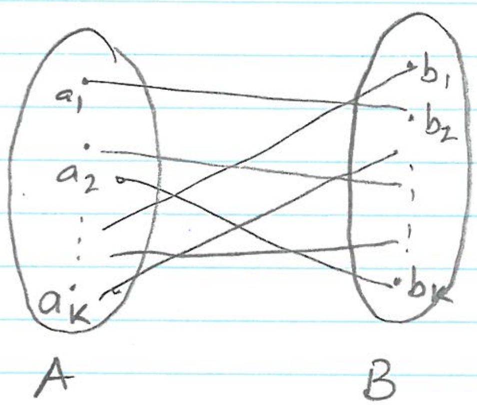
BIJECTION RULE

Let A and B are finite sets.

A function $f: A \to B$ is a bijection if

f is one-to-one : $\forall a \neq a'$ $f(a) \neq f(a')$

f is onto ; for each $b \in B$, $\exists$ an $a \in A$ such that $f(a) = b$.



→ Each element of A corresponds to a unique element of B

→ Each element of B corresponds to a unique element of A

⇒ If there exists a BIJECTION between two finite sets A and B then $|A| = |B|$.

Let S be a set of n items, where $n \geq 0$.

Let $A = \mathcal{P}(S)$ = Set of all subsets of S.

Let B = Set of all binary strings of length n.

We will establish a bijection $f : A \to B$ and then $|A| = |B| = 2^n$, as by product ~~rule~~ we know that $|B| = 2^n$.

First let us understand the set $\mathcal{P}(S)$.

Let $S = \{a, b\} \to \mathcal{P}(S) = \{\phi, \{a\}, \{b\}, \{a,b\}\}$

$S = \{a, b, c\} \to \mathcal{P}(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

Establishing a bijection between A and B.

Let $S = \{s_1, s_2, \cdots, s_n\}$ be the elements of S.

Consider a subset $T \subseteq S$.

Define a bit string $b_1 b_2 b_3 \cdots b_n$ of length n corresponding to T as follows

$$\forall i \in \{1..n\} \quad b_i = \begin{cases} 1 & \text{if } s_i \in T \\ 0 & \text{if } s_i \notin T \end{cases}$$

Example: Let $S = \{\overset{s_1}{a}, \overset{s_2}{b}, \overset{s_3}{c}, \overset{s_4}{d}\}$ then the corresponding bit strings will be

T	$b = b_1 b_2 b_3 b_4$
ϕ	0 0 0 0
T	1 1 1 1
$\{a, b\}$	1 1 0 0
$\{a, c\}$	1 0 1 0
$\{b, c, d\}$	0 1 1 1
⋮	⋮

Observe: there is 1-1 correspondance between bitstrings and subsets of S.

Each bitstring corresponds to a unique subset and each subset corresponds to unique bitstring.

$\Rightarrow$ If we define $f : A \to B$ as

$\forall\ T \in A$, (i.e T is a subset of S)

$f(T) = b_1 b_2 \ldots b_n$ where

$$b_i = \begin{cases} 1 & \text{if } s_i \in T \\ 0 & \text{if } s_i \notin T \end{cases}$$

then f is a bijection

Since $|B| = 2^n$

$\Rightarrow |A| = |\wp(S)| = 2^n$

COMPLEMENT RULE

$A \subseteq U$ then $|A| = |U| - |U \setminus A|$.

Problem: How many 4-letter words in English where a word can't contain all the letters that are the same.

Valid $\rightarrow$ {aabc, bcad, ccaf, ...}

Invalid $\rightarrow$ {aaaa, bbbb, ...}.

Note: We are just looking at words with lower case alphabets.

U = Set of all 4-letter words

A = Set of all 4-letter words, each word consisting of at least 2 distinct letters

$U \setminus A$ = Set of all 4-letter words, each word consists of the same letter.

$|U| = 26^4$ by product rule

$|U \setminus A| = 26 = |\{aaaa, bbbb, \ldots, zzzz\}|$

Thus, $|A| = |U| - |U \setminus A| = 26^4 - 26$
$= 456950$

Example 2: UNIX PASSWORDS

UNIX Passwords are 8 character long, where each of the character can be from

$$\mathcal{K} = \left\{ \begin{array}{l} ABCD \cdots \\ abcde \cdots \\ 1234567890 \\ !@\#\$ \cdots \cdot/? \end{array} \right\} \begin{array}{r} \to 26 \\ \to 26 \\ \to 10 \\ \to \underline{26} \\ 88 \end{array}$$

out of possible 128 ASCII characters.

How many possible passwords can be formed when at least one of the special characters has to be used?

U = Set of all 8-character words from alphabets in $\mathcal{K}$.

$|U| = 88^8$ by product rule.

A = Set of 8-character long valid passwords.

$U \setminus A$ = Set of 8 character long invalid passwords.

$\equiv$ Set of 8 character long strings composed only of upper-case alphabets, lower-case alphabets, and digits

$\equiv 62^8$

$\Rightarrow |A| = |U| - |U \setminus A| = 88^8 - 62^8 \sim 3.4 \times 10^{15}$

SUM-RULE

If $A_1, A_2, \ldots, A_m$ are pair-wise disjoint sets,
(i.e $A_i \cap A_j = \phi \ \forall \ 1 \le i < j \le m$)
then $|A_1 \cup A_2 \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$.

Problem: How many valid passwords of length atleast 6 (and atmost 8) are there in UNIX.

Define A_1 = Valid passwords of length 8
A_2 = Valid passwords of length 7
A_3 = Valid passwords of length 6.

Now $A_1 \cap A_2 = \phi$; $A_1 \cap A_3 = \phi$; $A_2 \cap A_3 = \phi$
as words are of different lengths.

Thus $|A_1 \cup A_2 \cup A_3| = A_1 + A_2 + A_3$

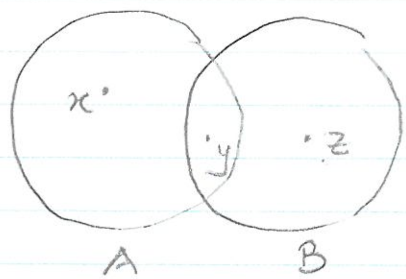
$$= (88^8 - 62^8) + (88^7 - 62^7) + (88^6 - 62^6).$$

INCLUSION - EXCLUSION PRINCIPLE

Let A and B be two finite-sets.

Then $|A \cup B| = |A| + |B| - |A \cap B|$.

Consider the Venn Diagram



Consider $|A| + |B|$

→ $x \in A$, $x \notin B$: x is counted once in $|A|+|B|$

→ $z \notin A$, $z \in B$: z is counted once in $|A|+|B|$.

→ $y \in A$, $y \in B$: y is counted twice in $|A|+|B|$

Thus, $|A \cup B| = |A| + |B| - |A \cap B|$.

Example : How many bit strings of length 8 either start with a 1 ~~bit~~ or end with 00.

Define A = Set of bit strings of length 8 starting with 1.

B = Set of bit strings of length 8 ending with 00.

$|A| = 2^7$ by product rule

$|B| = 2^6$ by product rule

$A \cap B$ = Set of bit strings of length 8 starting with a 1 and ending with 00.

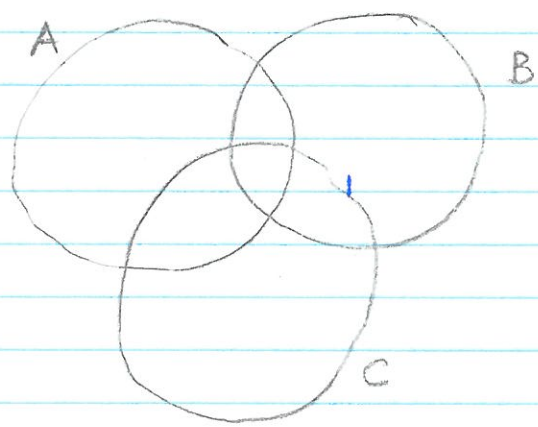
$|A \cap B| = 2^5$

Thus: $|A \cup B| = |A| + |B| - |A \cap B|$

$= 2^7 + 2^6 - 2^5 = 128 + 64 - 32$

$= 160$ bit-strings

Suppose we have three sets A, B, C.



Verify:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Example: Bit strings of length 8 having

either (a) 1 in first bit
or (b) 0 in last bit
or (c) 11 in 3rd & 4th bit from left

Define
Set of
A = Bit strings of length 8 starting with 1.
Set of
B = Bit strings of length 8 ending with 0.
C = Set of bit strings of length 8 having 11 in 3rd & 4th bit.

Observe that (by product rule)

$|A| = 2^7$

$|B| = 2^7$

$|C| = 2^6$

$|A \cap B| = 2^6$

$|A \cap C| = 2^5$

$|B \cap C| = 2^5$

$|A \cap B \cap C| = 2^4$

Thus

$$|A \cup B \cup C| = 2^7 + 2^7 + 2^6 - 2^6 - 2^5 - 2^5 + 2^4$$
$$= 206$$