# PERMUTATIONS

A permutation of a finite set $S$ is an ordered sequence of the elements of $S$, where each element occurs exactly once.

Let $S = \{a, b, c\}$

$S$ has 6 permutations: abc, acb, bac, bca, cab, cba.

e.g consider bac.
Each element of $S$ occurs exactly one.

**Claim:** Let $S$ be a set of $n \geq 0$ elements. There are exactly $n!$ permutations of $S$.

**Proof:** Apply product rule.

Procedure: Write down a permutation of $S$.

For $i = 1, 2, 3, \ldots, n$,

$i$-th task: Write down the $i$-th element in the permutation.

Task-1: $N_1 = n$ ways of performing

Task-2: $N_2 = n-1$, we can write any letter of $S$ except what was already written.

$i$-th-Task: $N_i = n - i + 1$
We can write any letter of $S$ except what was already written in first $i-1$ steps

Thus

$\#$ permutations $= \#$ ways to execute the $n$-tasks

$$= N_1 N_2 N_3 \cdots N_n$$

$$= n(n-1)(n-2) \cdots (n-n+1)$$

$$= n(n-1)(n-2) \cdots 1$$

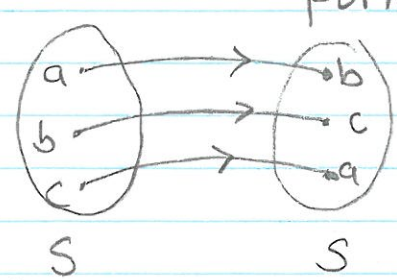
$$= n! \qquad \square$$

$$\boxed{\begin{array}{l} \text{Note that if } S = \phi; \quad |S| = 0 \\ \quad \text{and } S \text{ has } 0! = 1 \text{ permutations.} \end{array}}$$

Observe that a permutation of $S$ is a 1-1 function $f: S \to S$.

e.g if $S = \{a, b, c\}$ and let "bca" be a permutation. Then

$f$ :




i.e, $f(a) = b$; $f(b) = c$; $f(c) = a$

We have already counted $\#$ 1-1 functions $f: A \to B$, where $|A| = m$, $|B| = n$ as $\dfrac{n!}{(n-m)!}$

Since $|S| = n$, we obtain $\dfrac{n!}{0!} = n!$ many permutations.

# BINOMIAL COEFFICIENTS

$\binom{n}{k} \equiv$ # of k-element subsets of an n-element set

Example: $\binom{5}{3}$ = # of 3-element subsets of a 5-element set.

Let $S = \{a, b, c, d, e\}$.

3-element subsets are

$$\left.\begin{array}{l} \{a,b,c\},\ \{a,b,d\},\ \{a,b,e\} \\ \{a,c,d\},\ \{a,c,e\} \\ \{a,d,e\}. \\ \{b,c,d\},\ \{b,c,e\},\ \{b,d,e\} \\ \{c,d,e\} \end{array}\right\} \text{10 of them.}$$

Note that in terms of sets

$$\{a,b,c\} = \{c,b,a\} = \{a,c,b\} \ldots$$

Claim: $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$

Proof: Let $S$ be a set of $n$-elements.

Let $A$ be the set of all ordered sequences consisting of $k$ distinct elements of $S$.

We count # elements in $A$ in two different ways.

I: By Product Rule:

$$|A| = n(n-1)(n-2)\cdots(n-k+1)$$

$$|A| = \frac{n!}{(n-k)!} \qquad \text{— } 1$$

II: Using subsets:

Next, we write down all $\binom{n}{k}$ subsets of $S$ of size $k$ and then write down a list of all $k!$ permutations for each subset.

Thus $$|A| = \binom{n}{k} k! \qquad \text{— } 2$$

Equating 1 & 2 we obtain

$$\binom{n}{k} k! = \frac{n!}{(n-k)!} \quad \text{or} \quad \binom{n}{k} = \frac{n!}{k!\,(n-k)!} \quad \blacksquare$$

Examples:

1. How many different hands consisting of 4-cards from a 52-deck cards.

$$\binom{52}{4} = \frac{52!}{4!\,48!} = \frac{52 \cdot 51 \cdot 50 \cdot 49}{4 \cdot 3 \cdot 2 \cdot 1}$$

$$= 270725.$$

2. How many bit strings of length $n$ have exactly $k$ 1's, where $0 \le k \le n$.

Answer $\rightarrow \binom{n}{k}$

There is a bijection between

- set of all bit strings of length $n$ having exactly $k$ 1's
- set of $k$-element subsets of an $n$-element set.

3.

3. Membership Example

Consider a club consisting of 10 Women and 8 Men.

P1: How many ways a committee consisting of 5 people can be formed?

$\binom{18}{5}$ ≡ Choosing a set of 5 from a set of 18.

P2: Committee of 5 with exactly 3 Women.

Task 1: Choose 3 women out of 10 $\left(N_1 = \binom{10}{3}\right)$

Task 2: Choose 2 men out of 8. $\left(N_2 = \binom{8}{2}\right)$

By product rule

\# ways the committee can be formed

$$= N_1 N_2 = \binom{10}{3}\binom{8}{2} = \frac{10 \times 9 \times 8}{3 \times 2} \times \frac{8 \times 7}{2 \times 1}$$

$$= 3360$$

P3: Committee of 5 containing at least 3 Women.

Form three disjoint sets.

$C_3$: Committee consisting of 3 Women & 2 Men

$C_4$: Committee consisting of 4 Women & 1 Men

$C_5$: Committee consisting of 5 Women.

Thus

$$|C_3 \cup C_4 \cup C_5| = |C_3| \cup |C_4| \cup |C_5|$$

$$= \binom{10}{3}\binom{8}{2} + \binom{10}{4}\binom{8}{1} + \binom{10}{5}\binom{8}{0}$$

$$= 120 * 28 + 210 * 8 + 252 * 1$$

$$= 5292$$

P4: Committee ~~of~~ of 5 consisting of exactly 3 women but should not contain a pair (ALICE, BOB)

We will use complement rule.

Complement Rule $\quad |A| = |U| - |U \setminus A|$

$\hookrightarrow$ Committee consisting of 3 women & 2 men with no restrictions $\binom{10}{3}\binom{8}{2} \equiv 3360$

$|U \setminus A| \equiv$ Committee of 5 consisting of 3 women & 2 men and it includes (ALICE, BOB)
Since it includes ALICE & BOB, we need to find 2 women & 1 men to serve on the committee

$$= \binom{9}{2}\binom{7}{1} \equiv \frac{9 \times 8}{2} * 7 = 252$$

Thus $|A| = 3360 - 252 = 3208.$

# Newton's Binomial Theorem.

Consider

$$(x+y)^3 = (x+y)(x+y)(x+y)$$
$$= x^3 + 3x^2y + 3xy^2 + y^3$$

Each term is of degree 3 ; $x^i y^j$ such that $i+j=3$

There is a constant for each term.

Consider the second term: $3x^2y$

How is this formed from $(x+y)(x+y)(x+y)$ ?

By choosing $x$ from two of the terms and $y$ from the remaining terms.

How many ways we can choose 2 $x$'s from three terms : $\binom{3}{2} = 3$.

Consider
$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

and consider the term $10x^3y^2$.
How is $10x^3y^2$ formed from
$(x+y)(x+y)(x+y)(x+y)(x+y)$ ?

By choosing 3 $x$'s from the five terms.

In how many ways this can be done : $\binom{5}{3} = 10$.

Binomial Theorem.

For any integer $n \geq 0$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

→ Corollary: $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

Proof: Set $x=y=1$ in Binomial Theorem

→ Corollary: $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$ for $n \geq 1$.

Proof: Set $x=1$, $y=-1$ in Binomial Theorem

→ What is coefficient for $x^3y^4$ term in $(2x-3y)^7$ ?

Note that in $(a+b)^7$, the coefficient for $a^3b^4$ term is $\binom{7}{3}$.

The term is $\binom{7}{3} a^3 b^4 = \frac{7 \times 6 \times 5}{3 \times 2} a^3 b^4 = 35 a^3 b^4$

Replace a by $2x$ and b by $-3y$ and we obtain

$$35(2x)^3(-3y)^4 = 35 \cdot 2^2 \cdot (-3)^4 x^3 y^4 = 11340.$$