Permutations: An ordered sequence of elements of a finite set.

Binomial Coefficients:

$$\binom{n}{k} = \#\ k\text{-element subsets of an } n\text{-element set}$$

$$= \frac{n!}{k!\,(n-k)!}$$

Binomial Newton Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Combinatorial Proofs:

(a) $\binom{n}{k} = \binom{n}{n-k} \quad \forall\ 0 \le k \le n.$

(b) Pascal's: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

(c) Vandermonde's: $\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$

PASCAL'S TRIANGLE

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Combinatorial proofs : Use counting arguments.

1. $\forall\ 0 \leq k \leq n$,

$$\binom{n}{k} = \binom{n}{n-k}$$

Proof: $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$ and $\binom{n}{n-k} = \frac{n!}{(n-k)!\,(n-n+k)!}$

$$= \frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

Combinatorial proof: Let S be a set of n-elements

(I) $\binom{n}{k}$ = # of ways to choose k elements from S.

= # of ways to NOT choose (n-k)-elements from S

$= \binom{n}{n-k}$

(II) $\binom{n}{k} \equiv$ # of bit strings of length n consisting of exactly k 1's.

$\equiv$ # of bit strings of length n consisting of exactly n-k 0's.

$\equiv \binom{n}{n-k}$.

Pascal's Identity

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

Pf: Let S be a set of $n+1$ elements

$\binom{n+1}{k}$ = # sets of size k from S.

Two ways to count the same thing.

Fix an element $x \in S$.

Let $T = S \setminus \{x\}$; T is same as S except that $x \notin T$.

Note that $|T| = n$.

Consider any k element subset A of S.

Either

- it does not contain x

  [Then it is a k-element subset A of T,

  and there are $\binom{n}{k}$ such sets]

- it contains x

  [Suppose $A \subseteq S$, $|A| = k$ and $x \in A$.

  Consider $B = A \setminus \{x\}$.

  Note that $B \subseteq T$ and $|B| = k-1$.

Also, consider any subset $B \subseteq T$ such that $|B| = k-1$.

Construct $A = B \cup \{x\}$.
Note that $A \subseteq S$, $|A| = k$.

Thus, # k-elements subset of S containg x

$\equiv$ # $k-1$ elements subset of T.

Since # $k-1$ elements subset of $T = \binom{n}{k-1}$, we

Obtain that

k-elements subsets of S

$$\equiv \binom{n}{k} + \binom{n}{k-1}$$

subsets Not containg x

subsets. Contaiing x

$$= \binom{n+1}{k}$$

∎.

Vandermonde's Identity

For $m \geq 0$, $n \geq 0$, $m \geq r \geq 0$ and $n \geq r$

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}$$

Pf: Consider a set S with $m+n$ elements.

$$\binom{m+n}{r} = \#\ r\text{-subsets of } S.$$

Two ways

Let us count the number of r-subsets of S in an alternate way:

Let $A \subseteq S$, $B \subseteq S$ such that $A \cup B = S$ and $A \cap B = \phi$; i.e A and B partition S.

Consider a r-subset of S.

It may contain a k-subset of A and $(r-k)$-subset from B.

Define N_k for $0 \leq k \leq r$ as

Number of r-subsets of S that contain exactly k-elements from A (and $r-k$ elements from B)

Note that $\displaystyle\sum_{k=0}^{r} N_k = \binom{m+n}{r}$

How do we compute N_k?

Task 1 – choose k-elements of A
Task 2 – choose $(r-k)$-elements of B

By product rule,

$$N_k = (\#\text{ ways to do task } 1) * (\#\text{ ways to do task } 2)$$
$$= \binom{m}{k}\binom{n}{r-k}$$

as $|A| = m$ and $|B| = n$.

Thus
$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} \quad \blacksquare$$

<u>Corollary</u> $$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

Pf: Set $m = n = r$ in Vandermonde's Identity.

$$\binom{m+n}{r} = \binom{2n}{n}$$

$$\sum_{k=0}^{r} \binom{n}{k}\binom{n}{r-k} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} \qquad \binom{n}{n-k} = \binom{n}{k}$$

$$= \sum_{k=0}^{n} \binom{n}{k}\binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}^2 \quad \blacksquare$$

Pascal's Triangle

Recall Pascals Identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

and $\binom{n}{0} = 1$; $\binom{n}{n} = 1$.

Consider evaluation of $\binom{4}{3}$

$$\binom{4}{3} = \binom{3}{2} + \binom{3}{3}$$

$$= \binom{2}{1} + \binom{2}{2} + \binom{3}{3}$$

$$= \binom{1}{0} + \binom{1}{1} + \binom{2}{2} + \binom{3}{3}$$

$$= 1 + 1 + 1 + 1 .$$

Consider evaluation of $\binom{6}{3}$

$$\binom{6}{3} = \binom{5}{2} + \binom{5}{3}$$

$$= \binom{4}{1} + \binom{4}{2} + \binom{4}{2} + \binom{4}{3}$$

$$= \binom{3}{0} + \binom{3}{1} + \binom{3}{1} + \binom{3}{2} + \binom{3}{1} + \binom{3}{2} + \binom{3}{2} + \binom{3}{3}$$

$$= \binom{3}{0} + 2\binom{2}{0} + 2\binom{2}{1} + \binom{2}{1} + \binom{2}{2} + \binom{2}{0} + \binom{2}{1} + 2\binom{2}{1} + 2\binom{2}{2} + \binom{3}{3}$$

$$= \binom{3}{0} + 2\binom{2}{0} + 3\binom{2}{1} + \binom{2}{2} + \binom{2}{0} + 3\binom{2}{1} + 2\binom{2}{2} + \binom{3}{3}$$

$$= 1 + 2 + 3\binom{1}{0} + 3\binom{1}{1} + 1 + 1 + 3\binom{1}{0} + 3\binom{1}{1} + 2 + 1$$

$$= 1 + 2 + 3 + 3 + 1 + 1 + 3 + 3 + 2 + 1 \quad = 20$$

Arrange the binomial coefficients in a triangle where the n^{th} row contains the values

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots \binom{n}{n}.$$

Example:

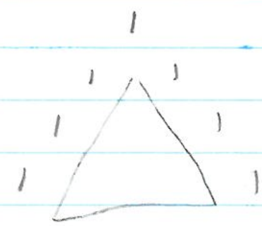
$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5} \to 1$$

0 row →
1 row →
2 row →
3 row →
~~4 row~~

$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

Observations:

(1) Each value along the boundary of Pascals Δ is 1.



2. Each value in the interior is sum of the two values above it $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

3. The values in the n^{th} row are equal to the coefficients in Newton Binomial Thm.

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

Values in 5^{th} row are $\begin{pmatrix} 1 & 5 & 10 & 10 & 5 & 1 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \searrow \\ \binom{5}{0} & \binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \binom{5}{4} & \binom{5}{5} \end{pmatrix}$

4. Sum of elements of the i^{th} row equals 2^i

e.g 5^{th} row $\equiv 1+5+10+10+5+1 = 32 = 2^5$

5. Each row is symmetric.

5^{th} row: 1 5 10 10 5 1 as $\binom{n}{k} = \binom{n}{n-k}$

6. Sum of squares of all values in n^{th} row equals the middle value in the $2n$-th row.

2^{nd} row: 1 2 1 Sum of Squares: $1+4+1 = 6$ $\rightarrow$ Middle value in 4^{th} row

How many different words can be made from

(a) WAWA

WAWA, WWAA, WAAW
AWWA, AWAW, AAWW

$\Big\}$ 6 in all.

Product Rule:

procedure → write down letters occuring the word

Task 1 → choose two positions out of 4 and write down W in those positions.

Tast 2 → In the remaining two positions write down in A.

$$N_1 = \binom{4}{2}$$

$$N_2 = 1$$

Thus # words from WAWA

$= \#$ ways to execute the procedure

$= N_1 N_2 = \binom{4}{2} \cdot 1 = 6$

(b) OTTAWA

Task 1: choose two positions out of 6 for A

$$N_1 = \binom{6}{2}$$

Task 2: choose two positions out of remaining 4 for T.

$$N_2 = \binom{4}{2}$$

Task 3: Choose a position out of remaining 2 for O.

$$N_3 = \binom{2}{1}$$

Task 4: In the remaining position write down W.

$$N_4 = 1$$

Thus # words from OTTAWA

$$= \binom{6}{2}\binom{4}{2}\binom{2}{1} 1 = 15 * 6 * 2$$
$$= 180 \blacksquare$$

Question:

How many positive integral solutions are there for $x_1 + x_2 + x_3 = 10$, where each $x_i \geq 0$.

Idea: Consider a solution for $x_1 + x_2 + x_3 = 10$.

Example: $x_1 = 3$; $x_2 = 2$; $x_3 = 5$

Consider a binary string of length 12 consisting of exactly two 1s.

0 0 0 1 0 0 1 0 0 0 0 0

Leading 0s will correspond to value of x_1.

zero's between two 1's will correspond to x_2

and trailing 0s will correspond to value of x_3.

Consider the two sets

$A = \{ (x_1, x_2, x_3) : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \text{ and } x_1 + x_2 + x_3 = 10 \}$

$B = \{$ Set of binary strings of length 12 consisting of exactly two 1s. $\}$

$f : A \to B$ is defined as follows:

Let $(x_1, x_2, x_3) \in A$

then $f(x_1, x_2, x_3)$ is a binary string of length 12 consiting of

x_1 # of 0's ; followed by a 1 ;

followed by x_2 # of 0's ; followed by a 1 ;

followed by x_3 # of 0's .

e.g: $f(3,4,3) = (000100001000)$

$f(3,0,7) = (000110000000)$

Observe: <u>f is 1 to 1</u>

if $(x_1, x_2, x_3) \neq (x_1', x_2', x_3')$ then

$$f(x_1, x_2, x_3) \neq f(x_1', x_2', x_3')$$

as at least one of $x_i \neq x_i'$ and thus the binary strings are not the same.

<u>f is onto</u>:

To show this we need to prove that for any bit string of length 12 with exactly two 1s, there is (x_1, x_2, x_3) such that

$x_1 + x_2 + x_3 = 10$ and $x_1 \geq 0$, $x_2 \geq 0$; $x_3 \geq 0$.

That is obvious from the interpretation of #0s before first 1, #0s between first & second 1, and trailing 0

Claim: $k \geq 1$, $n \geq 0$, $x_1 \geq 0$, $x_2 \geq 0$, $\ldots$, $x_k \geq 0$

solutions of equation $x_1 + x_2 + \cdots + x_k = n$

is $\binom{n+k-1}{k-1}$.

solution to inequality $x_1 + x_2 + \cdots + x_k \leq n$

is $\binom{n+k}{k}$.

Consider for example $x_1 + x_2 + x_3 \leq 10$

$\iff$ $x_1 + x_2 + x_3 + s = 10$ where $s = 10 - (x_1 + x_2 + x_3)$ and $s \geq 0$.

It is the same problem as before except that now we need to choose values of 4 variables. $\square$