Question:

How many positive integral solutions are there for $x_1 + x_2 + x_3 = 10$, where each $x_i \geq 0$.

Idea: Consider a solution for $x_1 + x_2 + x_3 = 10$.

Example: $x_1 = 3$; $x_2 = 2$; $x_3 = 5$

Consider a binary string of length 12 consisting of exactly two 1s.

0 0 0 1 0 0 1 0 0 0 0 0

Leading 0s will correspond to value of x_1.

zero's between two 1's will correspond to x_2

and trailing 0s will correspond to value of x_3.

Consider the two sets

$A = \{(x_1, x_2, x_3) : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \text{ and } x_1 + x_2 + x_3 = 10\}$

$B = \{$ Set of binary strings of length 12 consisting of exactly two 1s. $\}$.

$f : A \to B$ is defined as follows:

Let $(x_1, x_2, x_3) \in A$

then $f(x_1, x_2, x_3)$ is a binary string of length 12 consiting of

x_1 # of 0's; followed by a 1;

followed by x_2 # of 0's; followed by a 1;

followed by x_3 # of 0's.

e.g: $f(3,4,3) = (000100001000)$

$f(3,0,7) = (000110000000)$

Observe: <u>f is 1 to 1</u>

if $(x_1, x_2, x_3) \neq (x_1', x_2', x_3')$ then

$$f(x_1, x_2, x_3) \neq f(x_1', x_2', x_3')$$

as at least one of $x_i \neq x_i'$ and thus the binary strings are not the same.

<u>f is onto</u>?:

To show this we need to prove that for any bit string of length 12 with exactly two 1s, there is (x_1, x_2, x_3) such that

$x_1 + x_2 + x_3 = 10$ and $x_1 \geq 0$, $x_2 \geq 0$; $x_3 \geq 0$.

That is obvious from the interpretation of #0s before first 1, #0s between first & second 1, and trailing 0

Claim: $k \geq 1$, $n \geq 0$, $x_1 \geq 0$, $x_2 \geq 0$, ---, $x_k \geq 0$

solutions of equation $x_1 + x_2 + \cdots + x_k = n$

is $\binom{n+k-1}{k-1}$.

solution to inequality $x_1 + x_2 + \cdots + x_k \leq n$

is $\binom{n+k}{k}$.

Consider for example $x_1 + x_2 + x_3 \leq 10$

$\iff$ $x_1 + x_2 + x_3 + s = 10$ where $s = 10 - (x_1 + x_2 + x_3)$ and $s \geq 0$.

It is the same problem as before except that now we need to choose values of 4 variables. $\square$

PIGEON HOLE PRINCIPLE

Claim : Let $k \geq 1$ be an intyer. If $k+1$ or more objects are placed into k boxeo, then there is atleast one box containing two or more objects.

<u>Example 1</u> ; $f : A \to B$ such that $|A| > |B|$ cannot be one-to-one.

<u>Example 2</u> : Consider a set A of $(n+1)$ numbers from $\{1..2n\}$. Then there are two numbers in A such that one divides the other.

<u>Example 3</u> : Let $n \geq 1$ be an integer. Every sequence of n^2+1 distinct real numbers contains a subsequence of length $n+1$ that is either increasing or decreasing.

<u>Example 4</u> : What should be the smallest cleass size so that at least 16 students are guranteed to have the same grade among the 13-grade system $\{A^+, ---, D^-, F\}$ at Carleton.

<u>Example 5</u> : Show that a convex polyhedron has two faces with same number of sides.

Example 1:

Let A represent objects

Let B represent boxes.

Since # objects > # boxes $\Rightarrow \exists$ a box that contains two or more objects.

$\Rightarrow f$ is not one-to-one.

Example 2: Let $A = \{a_1, a_2, \cdots, a_{n+1}\}$, where each $1 \le a_i \le 2n$.

Express each $a_i = 2^{k_i} \cdot q_i$, where $k_i \geq 0$ is an integer and q_i is an odd integer.

e.g. $36 = 2^2 \cdot 9$

$47 = 2^0 \cdot 47$

$64 = 2^8 \cdot 1$

Each of $q_1, q_2, \cdots, q_{n+1}$ are odd and they belong to the set $\{1, 3, 5, 7, \cdots, 2n-1\} = A_{odd}$

Since $|A_{odd}| = n \Rightarrow \exists \; q_i, q_j \in \{q_1, q_2, \cdots, q_{n+1}\}$, $i \neq j$ such that $q_i = q_j$ by the Pigeon-hole principle.

Consider $a_i = 2^{k_i} \cdot q_i$ and $a_j = 2^{k_j} \cdot q_j$.

Thus, $\frac{a_i}{a_j} = \frac{2^{k_i} \cdot q_i}{2^{k_j} \cdot q_j} = 2^{k_i - k_j}$ as $q_i = q_j$.

If $k_i \geq k_j$ then a_j divides a_i,

If $k_j \geq k_i$ then a_i divides a_j. $\blacksquare$

Example 3: n^2+1 distinct numbers have either an increasing subsequence of length $\geq n+1$ or a decreasing subsequence of length $\geq n+1$.

Consider $10 = 3^2+1$ number

13, 7, 2, 15, 6, 5, 14, 3, 8, 11

e.g (2, 5, 8, 11) is an increasing subsequence of length ≥ 4.

<u>Proof</u>: Let $a_1, a_2, \ldots, a_{n^2+1}$ be the given numbers.

For each a_k associate a tupple (i_k, d_k), where $i_k \equiv$ length of longest increasing subsequence starting at a_k

Similarly, define

d_k = length of longest decreasing subsequence starting at a_k.

We need to show that there is an index $1 \le k \le n^2+1$ such that for that a_k, either $i_k \ge n+1$ or $d_k \ge n+1$.

We will prove this by contradiction.

Therefore assume that $i_k \le n$, $d_k \le n$ $\forall$ $1 \le k \le n^2+1$.

Consider the set

$$B = \{(b,c) \mid 1 \le b \le n,\ 1 \le c \le n\}$$

$$|B| = n^2.$$

Visualize each element of B as a box.

For each k, place the pair (i_k, d_k) at the same value in box.

Since $1 \le k \le n^2+1$, $\Rightarrow$ $\exists$ at least two pairs (i_s, d_s) and (i_t, d_t) such that

$1 \le s < t \le n^2+1$, and $i_s = i_t$, $d_s = d_t$ by the pigeon hole principle.

Consider a_s and a_t.

We know that $1 \le s < t \le n^2+1$.

Consider Case 1: $a_s < a_t$

then the length of increasing sequence at a_s is at least $1+i_t$, as we can take the increasing sequence starting at a_t and prefix it with a_s.

$$a_s \cdots \underbrace{a_t \cdots\cdots}_{i_t}$$

Hence $i_s \ge 1 + i_t$

$\Rightarrow i_s \ne i_t$ (a contradiction)

Consider Case 2: $a_s > a_t$

then by similar reasoning

$d_s \ge 1 + d_t \Rightarrow d_s \ne d_t$ (a contradiction)

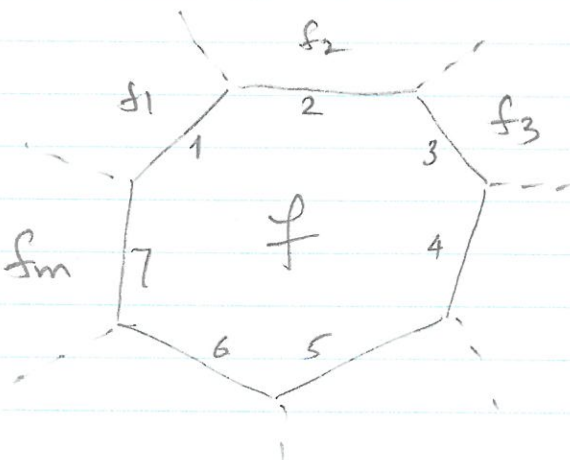
$\Rightarrow \exists$ a $1 \le k \le n^2+1$,

such either $i_k \ge n+1$ or $d_k \ge n+1$.

Example 5

Convex Polyhedron :

Consider the face with largest number of sides. Let it has m-sides



e.g $m=7$

Each of the adjacent faces has at least 3 sides and cannot have more than m-sides. Let f_1, f_2, ..., f_m be the adjacent faces.

In all we have $m+1$ faces in this group $\{f, f_1, f_2, \dots, f_m\}$ and each face has a degree between 3 and m.

Thus there are at least two faces of the same degree. ∎

Example 4: Consider a class of size

$16 * 13 + 1 \equiv 209$ students.

Claim: There are atleast 17 students obtaining the same grade.

If not, then the max # students in the class can be $\leq 16 * 13 \equiv 208$.

$$\boxed{\text{If } n \text{ objects are placed in } k\text{-boxes, then at least one box contains } \geq \lceil n/k \rceil \text{ objects.}}$$

Alternate formulation of pigeon hole principle.