RECURSION

An object (algorithm, function, set, ----) is defined by

- One or more base cases
- One or more rules that define the object in terms of smaller objects that are already well defined.

Examples :

1. $f : N \to N$ such that $f(0) = 2$; $f(n) = 2 \cdot f(n-1) + 1$ $\forall\, n \geq 1$.

2. Fibonacci Numbers $f_0 = 0$; $f_1 = 1$; $f_n = f_{n-1} + f_{n-2}$ $\forall\, n \geq 2$.

3. Rooted Binary Trees on n-nodes.

4. Countin binary strings of length n that do not contain 00.

5. A Gossip Problem : Sharing messages with minimum amount of communications.

6. Merge-Sort Algorithm

7. Counting max # regions when cutting a circle with n-cuts.

Example 1:

$f : \mathbb{N} \to \mathbb{N}$

$f(0) = 2$

$f(n) = 2 \cdot f(n-1) + 1 \quad \forall n \geq 1.$

Note that this function is well defined as

- $f(0)$ is uniquely defined
- $\forall n \geq 1$, if $f(n-1)$ is uniquely defined then $f(n) = 2 \cdot f(n-1) + 1$ is uniquely defined.

$\Rightarrow$ By induction, for any natural number n, the function value $f(n)$ is uniquely defined

Let us compute some of the values for f.

$n=0 \quad f(0) = 2$ [Base case]

$n=1 \quad f(1) = 2 \cdot f(0) + 1 = 2 \cdot 2 + 1 = 5$

(Apply recursion with $n=1$)

$n=2 \quad f(2) = 2 \cdot f(1) + 1 = 2 \cdot 5 + 1 = 11$

(apply recursion with $n=2$)

$n=3 \quad f(3) = 2 \cdot f(2) + 1 = 2 \cdot 11 + 1 = 23$

(apply recursion with $n=3$)

$n=4 \quad f(4) = 2 \cdot f(3) + 1 = 47$

Question: Can we express $f(n)$ in terms of n only?
(i.e. can we solve the recursion)

Claim

$$f(n) = 3 \cdot 2^{n} - 1 \quad \forall\, n \geq 0.$$

Proof: Proof by induction on $n \geq 0$.

Base case $n=0$ $f(0) = 3 \cdot 2^{0} - 1 = 3 - 1 = 2$ ✓

$n=1$ $f(1) = 3 \cdot 2 - 1 = 6 - 1 = 5$ ✓

I.H: Let $n \geq 2$ and assume it is true for $f(n-1)$.

i.e $f(n-1) = 3 \cdot 2^{n-1} - 1$.

Then,

$$f(n) = 2 \cdot f(n-1) + 1$$

$$= 2 \cdot \underbrace{\left[3 \cdot 2^{n-1} - 1\right]}_{\text{By I.H}} + 1$$

$$= 3 \cdot 2^{n} - 2 + 1 = 3 \cdot 2^{n} - 1 \;\blacksquare$$

Thus we have shown $f(n) = 3 \cdot 2^{n} - 1 \;\forall\, n \geq 0$.

Recursive definition of factorials.

$f : N \to N$

$f(0) = 1$

$f(n) = n \cdot f(n-1) \quad \forall \; n \geq 1.$

Show that $f(n) = n!$

Example 2: Fibonacci Numbers.

Two base case $f_0 = 0$; $f_1 = 1$;

and for $n \geq 2$ $\quad f_n = f_{n-1} + f_{n-2}$

Some of the elements of this famous sequences are

$$\underset{f_0}{0},\ \underset{f_1}{1},\ \underset{f_2}{1},\ \underset{f_3}{2},\ \underset{f_4}{3},\ \underset{f_5}{5},\ \underset{f_6}{8},\ \underset{f_7}{13},\ \underset{f_8}{21},\ \underset{f_9}{34},\ \underset{f_{10}}{55},\ \underset{f_{11}}{89},\ \ldots$$

Can we obtain a non-recursive formula?
i.e can we express f_n in terms of n?

Let $x^2 = x + 1 \iff x^2 - x - 1 = 0$

The two roots of this equation are

$$\varphi = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \frac{1-\sqrt{5}}{2} = \psi$$

$\approx 1.61 \qquad\qquad \approx -.61$

Claim: $f_n = \dfrac{\varphi^n - \psi^n}{\sqrt{5}} \quad \forall\, n \geq 0$

Pf: Base Case

$n=0 \quad f_0 = \dfrac{\varphi^0 - \psi^0}{\sqrt{5}} = \dfrac{1-1}{\sqrt{5}} = 0$

$n=1 \quad f_1 = \dfrac{\varphi^1 - \psi^1}{\sqrt{5}} = \dfrac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}}$

$= \dfrac{\sqrt{5}}{\sqrt{5}} = 1$

I.H: Let $n \geq 2$ and let claim be true for f_{n-1} and f_{n-2}.

i.e., $f_{n-1} = \dfrac{\varphi^{n-1} - \psi^{n-1}}{\sqrt{5}}$

$f_{n-2} = \dfrac{\varphi^{n-2} - \psi^{n-2}}{\sqrt{5}}$

Then $f_n = f_{n-1} + f_{n-2}$

$= \dfrac{\varphi^{n-1} - \psi^{n-1}}{\sqrt{5}} + \dfrac{\varphi^{n-2} - \psi^{n-2}}{\sqrt{5}} \quad \text{—} (*)$

observe that $\varphi^2 = \varphi + 1$ and $\psi^2 = \psi + 1$ as they are solution of equation $x^2 = x + 1$.

Then (*) can be rewritten as

$$f_n = \frac{\varphi^{n-1} + \varphi^{n-2}}{\sqrt{5}} - \frac{\psi^{n-1} + \psi^{n-2}}{\sqrt{5}}$$

$$= \frac{\varphi^{n-2}(\varphi + 1)}{\sqrt{5}} - \frac{\psi^{n-2}(\psi + 1)}{\sqrt{5}}$$

$$= \frac{\varphi^{n-2}\varphi^{2}}{\sqrt{5}} - \frac{\psi^{n-2}\psi^{2}}{\sqrt{5}}$$

$$= \frac{\varphi^{n}}{\sqrt{5}} - \frac{\psi^{n}}{\sqrt{5}} = \frac{\varphi^{n} - \psi^{n}}{\sqrt{5}} \quad \blacksquare$$

Thus

$$f(n) = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n}}{\sqrt{5}}.$$

Note that

$$\lim_{n \to \infty} \frac{f(n)}{f(n-1)} \longrightarrow \frac{1+\sqrt{5}}{2} = 1.61$$

Golden Ratio.

Counting binary strings of length n that do not contain 00.

Define B_n = # of bitstrings of length n that do not contain 00.

For example $B_3 = 5$

$\{010, 101, 111, 110, 011\}$

$B_4 = 8$

$\{1010, 0110, 0101, 1110, 1101, 1011, 0111, 1111\}$

$B_2 = 3$; $\{01, 10, 11\}$

If a bit string does not contain "00" – we call it valid.

Idea: For $n \geq 4$, we will express B_n in terms of B_{n-1} and B_{n-2}.

Consider the bit strings of length n that are valid strings. We will partition them in two groups.

Group 1: Bit strings in B_n that start with bit 1.

Since all these start with a 1, and do not contain 00, # of such strings equals to B_{n-1}.

= Valid strings of length $n-1$

Group 2: Valid strings in B_n that start with a 0-bit. In this case we know that the next bit has to be 1.

Thus # valid strings in B_n of group 2 $= B_{n-2}$.

Thus

$$B_n = B_{n-1} + B_{n-2} \quad \text{for } n \geq 4.$$

So we have

$$\begin{cases} B_2 = 3 \\ B_3 = 5 \\ B_n = B_{n-1} + B_{n-2} \quad \text{for } n \geq 4. \end{cases}$$

Thus the sequence for $n \geq 2$ is

$$\underset{B_2}{3}, 5, 8, 13, 21, 34, 55, 89, \ldots$$

B_3, B_4, B_5

This is same as the Fibonacci sequence, except that some initial terms are removed.

Observe that for $n \geq 2$,

$$B_n = f_{n+2}$$

n+2nd Fibonacci number.

A Gossip Problem

$n \geq 4$.

A group of n-people : $P_1, P_2, \ldots P_n$.

Each knows a scandal : $S_1, S_2, \ldots, S_n$

(P_i knows S_i for $1 \leq i \leq n$).

If P_i makes a phone call to P_j, then they exchange all the scandals they know with each other.

Goal: What is the minimum number of calls required so that everybody knows all the scandals.

Solution 1: Every pair P_i & P_j $1 \leq i < j \leq n$ makes a call. Observe that in the end everybody knows the scandal.

$$\# \text{ calls} = \frac{n(n-1)}{2}$$

Solution 2: Let $n = 4$.

Initially:

P_1	P_2	P_3	P_4
S_1	S_2	S_3	S_4

- P_1 calls P_2 and P_3 calls P_4

After the two calls:

P1	P2	P3	P4
$S_1 S_2$	$S_1 S_2$	$S_3 S_4$	$S_3 S_4$

- P_1 calls P_3 and P_2 calls P_4

After the 4th call we have

P1	P2	P3	P4
$S_1 S_2 S_3 S_4$	$S_1 S_2 S_3 S_4$	$S_1 S_2 S_3 S_4$	$S_1 S_2 S_3 S_4$

$\Rightarrow$ After 4 calls everybody knows the scandal, whereas Solution 1 will require $\frac{4*3}{2} = 6$ calls.

Consider in general for $n \geq 4$.

- We will assume that we know how to schedule calls for a group of $n-1$ people.
- We schedule the calls for a group of n as follows.

Initially: P_1 knows S_1, P_2 knows S_2, ..., P_n knows S_n.

Step1: P_{n-1} calls P_n. [P_1 knows S_1, P_2 knows S_2, ..., P_{n-2} knows S_{n-2} and P_{n-1} & P_n knows S_{n-1} & S_n]

Step2: Schedule calls for $\{P_1, P_2, ---, P_{n-1}\}$.
P_1 knows all scandals, P_2 knows all scandals, ---, P_{n-1} knows all scandals, P_n knows S_{n-1} & S_n.

Step3: P_{n-1} calls P_n. [Everybody knows all the scandals]

ALGORITHM GOSSIP(n):

// $n \geq 4$, Scheduling phone calls for $P_1, P_2, \ldots, P_n$

If $n = 4$
then
- P_1 calls P_2;
- P_3 calls P_4;

- P_1 calls P_3;
- P_2 calls P_4;

else
- P_{n-1} calls P_n;
- GOSSIP($n-1$);
- P_{n-1} calls P_n.

How many phone calls are made in Gossip(n)?

Let $C(n)$ = # phone calls made by Gossip(n).

We know $C(4) = 4$. and

for $n \geq 5$,

$$C(n) = 2 + C(n-1).$$

Our recurrence is

$$C(4) = 4 \quad \text{and} \quad C(n) = 2 + C(n-1) \text{ for } n \geq 5.$$

eg $C(4)=4$, $C(5) = 2+4=6$; $C(6) = 2+6=8$; ....

$C(7) = 2+8=10$, ...

thus $C(n) = 2n-4$ for $n \geq 4$.

This can be shown by induction

as $C(4) = 2 \cdot 4 - 4 = 4$ (base case)

assume it is true for $n-1$ for $n \geq 5$,

i.e $C(n-1) = 2(n-1) - 4$,

Then, $C(n) = 2 + C(n-1)$

$$= 2 + \{2(n-1) - 4\}$$

$$= 2 + 2n - 2 - 4 = 2n - 4 \quad \blacksquare$$

It turns out $2n-4$ calls are necessary!