COMP 2804 — Assignment 1
Solutions

Question 1: On the first page of your assignment, write your name, student number, and course number COMP 2804.

Solution:
Name: Luke Skywalker
Student Number: 100001138
Course Number: COMP 2804

Question 2: Consider a restricted alphabet set that is made of letters $\{A, C, T, G\}$. How many 5-letter words you can form from $\{A, C, T, G\}$ with the following restrictions (please justify your answer):

1. The word ends with A.
2. The word starts with A and ends with T.
3. The word contains only C and G.
4. The word does not contain C.
5. The word does not contain AAA in consecutive positions.
(For example, CTAAA, TAAAC, AAAAC are not valid.)

Solution:

1. For the first four letters of the word we have 4 options (A,C,T,G). Since the word ends with A, we only have one choice (A) for the last letter. Thus, by the product rule, there are $4^4(1) = 256$ words that end with A.
2. We have 1 option for the first letter (A), 1 option for the last letter (T) and 4 options for the remaining 3 letters. By applying the product rule we see that there are $(1)(4^3)(1) = 64$ words that start with A and end with T.
3. We only have 2 options (C,G) for each of the 5 letters. Therefore, by the product rule, there are $2^5 = 32$ words that contain only C and G.
4. Since the word cannot contain C, we have 3 options (A,T,G) for each of the 5 letters. Thus, by the product rule, there are $3^5 = 243$ words that do not contain C.
5. We will use the complement rule here. We consider the total number of possible 5-letter words over our alphabet and subtract off the words that contain AAA in consecutive

positions.

Let P be the set of all 5-letter words that do not contain AAA in consecutive positions.
Let U be the set of all 5-letter words over our alphabet.
By the product rule, there are $4^5 = 1024$ possible 5-letter words, since there are 4 possibilities for each of the 5 letters. Thus,

$$|U| = 4^5 = 1024$$

Now, we count the number of possible 5-letter words that do contain AAA.
Let S be the set of 5-letter words that contain "AAA" (Note that $S = \{U \setminus P\}$).
We note that there are three ways for a string to contain "AAA": AAA**, *AAA*, and **AAA, where * is some letter from our alphabet.

Let S_1 be the set of 5-letter words of the form "AAA**"
S_2 be the set of 5-letter words of the form "*AAA*"
S_3 be the set of 5-letter words of the form "**AAA"

Using the sum rule and the principle of inclusion-exclusion:

$$|S| = |S_1| + |S_2| + |S_3| - |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| + |S_1 \cap S_2 \cap S_3|$$

- For $|S_1|$: We only have 1 option (A) for the first 3 letters. We have 4 options for the last 2 letters. By the product rule $|S_1| = 4^2 = 16$.
- For $|S_2|$: We have 4 options for the first and last letters. We only have 1 option (A) for the remaining 3 letters. By the product rule $|S_2| = 4^2 = 16$.
- For $|S_3|$: We have 4 options for the first 2 letters. We only have 1 option (A) for the remaining 3 letters. By the product rule, $|S_3| = 4^2 = 16$.
- For $|S_1 \cap S_2|$: A word in this set is of the form: "AAAA*". We only have 1 option (A) for the first 4 letters. There are 4 options for the last letter. By the product rule, $|S_1 \cap S_2| = 4$.
- For $|S_1 \cap S_3|$: A word in this set is of the form: "AAAAA". There is only 1 word that satisfies this constraint, namely the word AAAAA. Hence, $|S_1 \cap S_3| = 1$.
- For $|S_2 \cap S_3|$: A word in this set is of the form: "*AAAA". We only have 1 option (A) for the last 4 letters. There are 4 options for the first letter. By the product rule, $|S_2 \cap S_3| = 4$.
- For $|S_1 \cap S_2 \cap S_3|$: There is only 1 word in this set: AAAAA. Thus, $|S_1 \cap S_2 \cap S_3| = 1$.

Altogether,

$$\begin{aligned}
|S| &= |S_1| + |S_2| + |S_3| - |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| + |S_1 \cap S_2 \cap S_3| \\
&= 16 + 16 + 16 - 4 - 4 - 1 + 1 \\
&= 40
\end{aligned}$$

Now, applying the compliment rule

$$\begin{aligned} |P| &= |U| - |U \setminus P| \\ &= |U| - |S|, \text{ since } \{U \setminus P\} = S \\ &= 1024 - 40 \\ &= 984 \end{aligned}$$

Therefore, there are 984 words that do not contain AAA in consecutive positions.

Question 3: Let $n > 0$ be even. A binary string B is called a palindrome, if you obtain the same string, either you read the bits from left to right or from right to left. For example, the strings 001100 and 101101 are palindromes, whereas 010100 and 110001 are not palindromes. How many bit strings of length n are palindromes?

Solution: To form a palindrome of length n we can simply take a bit string b of length $n/2$ and find its reverse bit string, b'. (Note that n is divisible by 2 since it is even). Now we simply append the two bit strings, giving us bb', which will be a palindrome. Thus, counting the number of bit strings of length n that are palindromes is equivalent to counting the number of bit strings of length $n/2$. Therefore, by the product rule, there are $2^{n/2}$ bit strings of length n that are palindromes.

Question 4: A polygon on n-vertices is convex, if the segment joining any pair of points, either on the boundary or in the interior of the polygon, lies entirely within the polygon. A diagonal in a polygon is a line segment connecting a pair of non-adjacent vertices. How many diagonals does a convex polygon on n vertices can have?

Solution: Each of the n vertices can be connected to $n-3$ other, non-adjacent vertices to form a diagonal. If we count like this, we count each diagonal twice, once for each endpoint. Hence we must divide by 2. Therefore, there are $\frac{n(n-3)}{2}$ diagonals in a convex polygon with n vertices.

Question 5: A coin is flipped 8 times, and in each flip the outcome is either **H**ead or **T**ail. How many outcomes have:

1. Exactly four Heads?
2. At most four Heads?
3. Same number of Heads and Tails?
4. More Heads than Tails?

Solution: We note that there is a bijection between the set of outcomes that have exactly k Heads when a coin is flipped n times and the set of all bit strings of length n that have exactly k 1s. In class, we saw that there are $\binom{n}{k}$ bit strings of length n that have exactly k 1s. Hence by the bijection rule, when a coin is flipped n times, the number of outcomes with exactly k Heads is $\binom{n}{k}$. In our problem the coin is flipped 8 times, and hence our $n = 8$.

1. Here $k = 4$, thus $\binom{8}{4}$ outcomes have exactly 4 Heads.

2. Finding the number of outcomes that have four Heads is equivalent to summing the number of outcomes that have exactly 0 Heads, exactly 1 Head, exactly 2 Heads, exactly 3 Heads and exactly 4 Heads (note that these sets are pairwise disjoint). Using the sum rule we obtain the following:

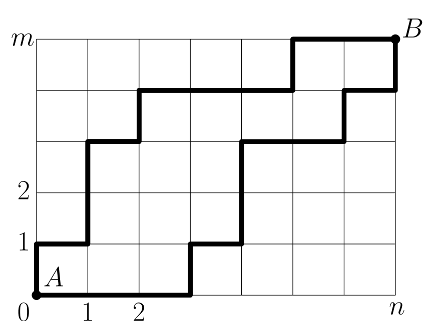
$$\begin{aligned}
\text{at most 4 H} &= \text{exactly 0 H} + \text{exactly 1 H} + \text{exactly 2 H} + \text{exactly 3 H} + \text{exactly 4 H} \\
&= \binom{8}{0} + \binom{8}{1} + \binom{8}{2} + \binom{8}{3} + \binom{8}{4} \\
&= 1 + 8 + 28 + 56 + 70 \\
&= 163
\end{aligned}$$

3. If an outcome has the same number of Heads and Tails, then there are 4 Heads and 4 Tails and hence exactly 4 Heads. This is the same as part 1, and thus there are $\binom{8}{4}$ outcomes that have the same number of Heads and Tails.

4. If an outcome has more Heads than Tails, then there are either exactly 5 Heads, exactly 6 Heads, exactly 7 Heads or exactly 8 Heads. We note that these are pairwise disjoint sets. Using the sum rule we get:

$$\begin{aligned}
\text{more H than T} &= \text{exactly 5 H} + \text{exactly 6 H} + \text{exactly 7 H} + \text{exactly 8 H} \\
&= \binom{8}{5} + \binom{8}{6} + \binom{8}{7} + \binom{8}{8} \\
&= 56 + 28 + 8 + 1 \\
&= 93
\end{aligned}$$

Question 6:



Let $n > 0$ and $m > 0$ be integers. Consider an $n \times m$ rectangle as shown in figure. We are interested in finding paths between the point $A = (0,0)$ and $B = (n,m)$. A path starts at $A = (0,0)$ and in each step, we are allowed to move one unit either in $+x$-direction or in $+y$-direction, till we reach $B = (n,m)$. Such type of paths are called monotone paths. (Examples of two such paths are shown in bold in the figure.) Note that any monotone path is composed of exactly n unit horizontal segments and m unit vertical segments. How many monotone paths are there from A to B? Justify your answer.

Solution: A path from A to B can be expressed as a string of directions. If we let v represent a move one unit in the $+x$-direction and h represent a move one unit in the $+y$-direction, then we can create a string of length $n+m$ that includes n vs and m hs. For example, the paths in the example diagram would be represented by: $\{v,h,v,v,h,v,h,h,h,v,h,h\}$ and $\{h,h,h,v,h,v,v,h,h,v,h,v\}$. Thus, there is a bijection between the paths from A to B and the strings of length $n+m$ that include n vs and m hs. There is also a bijection between the strings of length $n+m$ that include n vs and m hs and the set of bit strings of length $n+m$ with n 1s (simply replace every v with a 1 and every h with a 0). By the bijection rule, the number of monotone paths is then equal to the number of bit strings of length $n+m$ with n 1s. There are $\binom{n+m}{n}$ such bit strings and hence there are $\binom{n+m}{n}$ monotone paths from A to B.

Question 7: We need to select a committee consisting of n members from a group n women and n men, such that the chairperson of the committee must be a woman. By counting in two different ways, the total number of ways such a committee can be formed, show that

$$n\binom{2n-1}{n-1} = \sum_{r=1}^{n} r\binom{n}{r}^2.$$

Solution:
First way:

- First task: Pick the chairperson of the committee. There are n ways to do this.
- Second task: Pick the rest of the committee. There are $\binom{2n-1}{n-1}$ ways to do this.

By the Product Rule, the number of ways to select the committee is equal to

$$n\binom{2n-1}{n-1} \tag{1}$$

Second way:
Let A be the set of all committees consisting of n members from a group of n women and n men where the chairperson of the committee is a woman.
We define the following sets: A_1 is the set of all committees that has exactly 1 woman and

$n-1$ men, A_2 is the set of all committees that has exactly 2 women and $n-2$ men, ... , A_{n-1} is the set of all committees that has exactly $n-1$ women and 1 man, A_n is the set of all committees that has exactly n women and 0 men.
Since all of these sets are pairwise disjoint, we apply the sum rule:

$$\begin{aligned}|A| &= |A_1| + |A_2| + \ldots + |A_{n-1}| + |A_n| \\ &= \sum_{r=1}^{n} |A_r|\end{aligned}$$

To find $|A_r|$:

- First task: Pick r women from the set of n women. There are $\binom{n}{r}$ ways to do this.
- Second task: Pick the chairperson. There are r ways to do this.
- Third task: Pick $n-r$ men from the set of n men. There are $\binom{n}{n-r}$ ways to do this.

By the Product Rule,

$$\begin{aligned}|A_r| &= r\binom{n}{r}\binom{n}{n-r} \\ &= r\binom{n}{r}\left(\frac{n!}{(n-r)!\cdot r!}\right) \\ &= r\binom{n}{r}\left(\frac{n!}{r!\cdot(n-r)!}\right) \\ &= r\binom{n}{r}\binom{n}{r} \\ &= r\binom{n}{r}^2\end{aligned}$$

Then altogether we have:

$$\begin{aligned}|A_r| &= \sum_{r=1}^{n} |A_r| \\ &= \sum_{r=1}^{n} r\binom{n}{r}^2 \qquad (2)\end{aligned}$$

The values of (1) and (2) must be equal since both of the count the number of ways to form a committee consisting of n members from a group of n women and n men, such that the chairperson of the committee must be a woman.

Question 8: Assume $x \neq 0$. What is the coefficient of x^1 and x^2 in the expansion of

$$\frac{(1+x)^n - 1}{x}$$

Solution: First, we look at the expansion of $(1+x)^n$. Using the binomial theorem, we get that

$$\begin{aligned}(1+x)^n &= \binom{n}{0}x^0 + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + \binom{n}{n}x^n \\ &= 1 + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + \binom{n}{n}x^n\end{aligned}$$

Plugging this into our full equation gives us:

$$\begin{aligned}\frac{(1+x)^n - 1}{x} &= \frac{\binom{n}{0}x^0 + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + \binom{n}{n}x^n - 1}{x} \\ &= \frac{1 + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + \binom{n}{n}x^n - 1}{x} \\ &= \frac{\binom{n}{1}x^1 + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + \binom{n}{n}x^n}{x} \\ &= \binom{n}{1} + \binom{n}{2}x^1 + \binom{n}{3}x^2 + \ldots + \binom{n}{n}x^{n-1}\end{aligned}$$

Therefore the coefficient of x^1 is $\binom{n}{2}$ and the coefficient of x^2 is $\binom{n}{3}$ in the expansion.

Question 9: Consider the following equation

$$(1+x)^{n+1} - 1 = x(1 + (1+x) + (1+x)^2 + \cdots + (1+x)^n)$$

1. Show that the equation holds for all positive values of n.
   For example you can prove this by using induction on n.

2. By looking at coefficients of the term x^2 in the equation, show that

$$\binom{n+1}{2} = \binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \cdots + \binom{n}{1}.$$

3. Derive a similar type of expression by looking at the coefficients of the term x^3 in the equation, i.e., is it true that

$$\binom{n+1}{3} = \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n}{2}.$$

Solution:

1. We will prove $(1+x)^{n+1} - 1 = x(1+(1+x)+(1+x)^2+\cdots+(1+x)^n)$ holds for all $n \geq 1$ using a proof by induction on n.
   Base Case: $n = 1$
   LHS:
   $$\begin{aligned}(1+x)^{n+1} - 1 &= (1+x)^{(1)+1} - 1 \\ &= (1+x)^2 - 1 \\ &= x^2 + 2x + 1 - 1 \\ &= x^2 + 2x\end{aligned}$$
   RHS:
   $$\begin{aligned}x(1+(1+x)^n) &= x(1+(1+x)^{(1)}) \\ &= x(2+x) \\ &= 2x + x^2\end{aligned}$$
   LHS = RHS and thus our base case holds.

   Inductive Hypothesis: Assume $(1+x)^{n+1} - 1 = x(1+(1+x)+(1+x)^2+\cdots+(1+x)^n)$ holds for all $n \geq 1$.
   Inductive Step: We will now show that it holds for $n+1$; we want to show $(1+x)^{(n+1)+1} - 1 = x(1+(1+x)+(1+x)^2+\cdots+(1+x)^{n+1})$.
   Starting with the RHS:
   $$\begin{aligned}RHS &= x(1+(1+x)+(1+x)^2+\cdots+(1+x)^{n+1}) \\ &= x(1+x)^{n+1} + x(1+(1+x)+(1+x)^2+\cdots+(1+x)^n) \\ &= x(1+x)^{n+1} + (1+x)^{n+1} - 1, \text{ by applying the inductive hypothesis} \\ &= (1+x)(1+x)^{n+1} - 1 \\ &= (1+x)^{(n+1)+1} - 1 \\ &= LHS\end{aligned}$$
   Thus, $(1+x)^{(n+1)+1} - 1 = x(1+(1+x)+(1+x)^2+\cdots+(1+x)^{n+1})$.

   Therefore, we have shown by a proof by induction that the equation holds for all positive values of n.

2. First we look at the LHS of the original equation. Using the binomial theorem, the coefficient of the x^2 term is $\binom{n+1}{2}$.

Looking at the RHS, since we are multiplying $1+(1+x)+(1+x)^2+\cdots+(1+x)^{n+1}$ by x, we care about the coefficients of x^1 in the expansions of $(1+x)$, $(1+x)^2$, $(1+x)^3, \ldots,$ $(1+x)^n$. By the binomial theorem, the coefficients of the x^1 terms are $\binom{1}{1}, \binom{2}{1}, \binom{3}{1}, \ldots, \binom{n}{1}$ respectively. As in the original equation, we sum these coefficients to obtain the overall coefficient of the x^2 term, giving us $\binom{1}{1}+\binom{2}{1}+\binom{3}{1}+\cdots+\binom{n}{1}$.

Since the coefficient of the x^2 term in the LHS must equal the coefficient of the x^2 term in the RHS, we see that $\binom{n+1}{2}=\binom{1}{1}+\binom{2}{1}+\binom{3}{1}+\cdots+\binom{n}{1}$.

3. In a similar manner to part 2, first we look at the LHS of the original equation. Using the binomial theorem, the coefficient of the x^3 term is $\binom{n+1}{3}$.

   Looking at the RHS, since we are multiplying $1+(1+x)+(1+x)^2+\cdots+(1+x)^{n+1}$ by x, we care about the coefficients of x^2 in the expansions of $(1+x)^2$, $(1+x)^3$, $(1+x)^4, \ldots,$ $(1+x)^n$. By the binomial theorem, the coefficients of the x^1 terms are $\binom{2}{2}, \binom{3}{2}, \binom{4}{2}, \ldots, \binom{n}{2}$ respectively. As in the original equation, we sum these coefficients to obtain the overall coefficient of the x^2 term, giving us $\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\cdots+\binom{n}{2}$.

   Since the coefficient of the x^3 term in the LHS must equal the coefficient of the x^2 term in the RHS, we see that $\binom{n+1}{3}=\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\cdots+\binom{n}{2}$.

Question 10: How many ways you can fill your bag with 12 fruits from the Caribbean fruit store which sells four types of fruits (Mango, Sugar-Apple, Jamun, and Guava)? You can assume that the store has more than 12 fruits of each type.

Solution: For each of the 12 fruits we put in our bag, we have 4 options (Mango, Sugar-Apple, Jamun, and Guava). Using the product rule we see that there are $4^{12}=16777216$ ways to fill our bag with delicious fruit.

Alternatively, if you do not care about the order in which the fruits are placed in the bag, then the problem reduces to finding the number of possible integral solutions of the equation

$$x_1+x_2+x_3+x_4=12,$$

where each $x_i \geq 0$, and x_1, x_2, x_3, and x_4 represent the number of Mango, Sugar-Apple, Jamun, and Guava, respectively. These solutions have one to one correspondence to the bit sequences consisting of 15-bits, where each sequence has exactly 12 zero-bits and 3 one-bits. The number of these bit-sequences are $\binom{15}{3}$. (Refer to the class notes or Michiel's course notes for more details.)