0.1 Section Title

Let $n \geq 1$ and $k \geq 0$ be integers. Consider n people $P_1, P_2, \ldots, P_n$ who are outside of a room. Inside this room, there are $n+k$ chairs $C_1, C_2, \ldots, C_{n+k}$. These n people enter the room one by one in the order of their indices and, for each i with $1 \leq i \leq n$, person P_i is supposed to sit down in chair C_i. However, the first person P_1 is drunk[1] and, instead of taking chair C_1, chooses one of the $n+k$ chairs uniformly at random and sits down in the chosen chair. (Chair C_1 may be the chosen chair.) From then on, when person P_i enters the room, P_i checks if chair C_i is occupied. If this is not the case, P_i sits down in chair C_i. Otherwise, P_i chooses one of the unoccupied chairs uniformly at random and sits down in the chosen chair.

We want to determine the probability that, at the end, the last person P_n sits in chair C_n. Before we analyze this probability, we present the algorithm in pseudocode:

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Algorithm TakeASeat(n, k):

    // the input consists of n persons P_1, P_2, ..., P_n and
    // n + k chairs C_1, C_2, ..., C_{n+k}
    // all random choices made are mutually independent
    j = uniformly random element of {1, 2, ..., n + k};
    person P_1 sits down in chair C_j;
    for i = 2 to n
    do if chair C_i is unoccupied
         then person P_i sits down in chair C_i
         else j = index of a uniformly random unoccupied chair;
                person P_i sits down in chair C_j
         endif
    endfor
```

We define the event

$$A_{n,k} = \text{"after algorithm TakeASeat}(n,k) \text{ has terminated, person } P_n \text{ sits in chair } C_n\text{".}$$

The probability that was mentioned above is given by

$$p_{n,k} = \Pr(A_{n,k}).$$

[1] maybe P_1 is the President of the Carleton Computer Science Society?

We will present two different ways to determine the probability $p_{n,k}$.

0.1.1 A First Solution

Let us start with the case when $n = 2$. In this case, event $A_{2,k}$ happens if and only if person P_1 chooses one of the chairs $C_1, C_3, C_4, \ldots, C_{2+k}$. Since P_1 chooses one of the $2 + k$ chairs uniformly at random, it follows that

$$p_{2,k} = \Pr(A_{2,k}) = \frac{k+1}{k+2}.$$

Assume from now on that $n \geq 3$. We are going to derive a recurrence relation that expresses $p_{n,k}$ in terms of $p_{2,k}, p_{3,k}, \ldots, p_{n-1,k}$.

Consider the index j of the chair that P_1 chooses.

- If $j \in \{1, n+1, n+2, \ldots, n+k\}$, then for each $i = 2, 3, \ldots, n$, chair C_i is unoccupied at the beginning of the i-th iteration and, thus, person P_i sits down in chair C_i. In particular, event $A_{n,k}$ occurs.

- If $j = n$, then chair C_n is occupied at the beginning of the n-th iteration and, thus, event $A_{n,k}$ does not occur.

- Assume $j \in \{2, 3, \ldots, n-1\}$. Then for each $i = 2, 3, \ldots, j-1$, chair C_i is unoccupied at the beginning of the i-th iteration and, thus, person P_i sits down in chair C_i. At the beginning of the j-th iteration, the chairs $C_1, C_{j+1}, C_{j+2}, \ldots, C_{n+k}$ are unoccupied and person P_j chooses one of these chairs uniformly at random. Thus, the iterations $j, j+1, \ldots, n$ can be viewed as running algorithm TakeASeat$(n-j+1, k)$, where the $n-j+1$ persons are $P_j, P_{j+1}, \ldots, P_n$ and the $n-j+1+k$ chairs are $C_1, C_{j+1}, C_{j+2}, \ldots, C_{n+k}$. In this case, event $A_{n,k}$ occurs if and only if, after algorithm TakeASeat$(n-j+1, k)$ has terminated, person P_n sits in chair C_n, i.e., event $A_{n-j+1,k}$ occurs.

Based on this, we define the events

$$B_{n,k,j} = \text{"during algorithm TakeASeat}(n,k)\text{, person } P_1 \text{ chooses chair } C_j\text{"}$$

for $j = 1, 2, \ldots, n+k$. Since exactly one of these events is guaranteed to occur, we can apply the Law of Total Probability (Theorem **??**) and obtain

$$\Pr(A_{n,k}) = \sum_{j=1}^{n+k} \Pr(A_{n,k} \mid B_{n,k,j}) \cdot \Pr(B_{n,k,j}).$$

For any j with $1 \leq j \leq n+k$, we have $\Pr(B_{n,k,j}) = 1/(n+k)$. We have seen above that

$$\Pr(A_{n,k} \mid B_{n,k,j}) = \begin{cases} 1 & \text{if } j \in \{1, n+1, n+2, \ldots, n+k\}, \\ 0 & \text{if } j = n, \\ \Pr(A_{n-j+1,k}) & \text{if } j \in \{2, 3, \ldots, n-1\}. \end{cases}$$

We conclude that

$$\begin{aligned} p_{n,k} &= \Pr(A_{n,k}) \\ &= \frac{k+1}{n+k} + \sum_{j=2}^{n-1} \Pr(A_{n-j+1,k}) \cdot \Pr(B_{n,k,j}) \\ &= \frac{k+1}{n+k} + \sum_{j=2}^{n-1} p_{n-j+1,k} \cdot \frac{1}{n+k}, \end{aligned}$$

i,e,. we have derived the recurrence relation for $p_{n,k}$. If we write out the terms in this summation, then we get, for $n \geq 3$,

$$p_{n,k} = \frac{k+1}{n+k} + \frac{1}{n+k}\left(p_{2,k} + p_{3,k} + \cdots + p_{n-1,k}\right).$$

As we have seen above, the base case is given by

$$p_{2,k} = \frac{k+1}{k+2}.$$

It remains to solve the recurrence. If you use the recurrence to determine $p_{n,k}$ for some small values of n, then you will notice that they are all equal to $(k+1)/(k+2)$. This suggests that

$$p_{n,k} = \frac{k+1}{k+2}$$

for all integers $n \geq 2$. Using induction on n, it can easily be proved that this is indeed the case. Note that $p_{n,k}$ does not depend on n. In particular, if $k = 0$, then the probability that person P_n sits in chair C_n is equal to $1/2$.

0.1.2 A Second Solution

Our second solution is obtained by modifying algorithm TAKEASEAT$(, n, k)$: Person P_1 is still drunk and, instead of taking chair C_1, chooses one of the

$n+k$ chairs uniformly at random and sits down in the chosen chair. From then on, for $i = 2, 3, \ldots, n-1$, when person P_i enters the room, P_i checks if chair C_i is occupied. If this is not the case, P_i sits down in chair C_i. Otherwise, P_i kicks P_1 out of chair C_i, P_i sits down in chair C_i, after which P_1 chooses one of the unoccupied chairs uniformly at random and sits down in the chosen chair. In pseudocode, this new algorithm is as follows:

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Algorithm TakeASeat'(n, k):

    // the input consists of n persons P_1, P_2, ..., P_n and
    // n + k chairs C_1, C_2, ..., C_{n+k}
    // all random choices made are mutually independent
    j = uniformly random element of {1, 2, ..., n + k};
    person P_1 sits down in chair C_j;
    for i = 2 to n − 1
    do // P_2 sits in C_2, P_3 sits in C_3, ..., P_{i−1} sits in C_{i−1}
        if chair C_i is occupied
        then // P_1 is in C_i
                j = uniformly random element of {1, i + 1, i + 2, ..., n + k};
                person P_1 sits down in chair C_j
        endif;
        person P_i sits down in chair C_i
    endfor
```

After algorithm TakeASeat$'(n,k)$ has terminated, each person P_i (for $i = 2, 3, \ldots, n-1$) sits in chair C_i, whereas person P_1 sits in one of the chairs $C_1, C_n, C_{n+1}, \ldots, C_{n+k}$. Event $A_{n,k}$ occurs if and only if, again after algorithm TakeASeat$'(n,k)$ has terminated, person P_1 sits in one of the chairs $C_1, C_{n+1}, C_{n+2}, \ldots, C_{n+k}$. Intuitively, at the end of the algorithm, P_1 sits in a uniformly random chair from the $k+2$ chairs $C_1, C_n, C_{n+1}, \ldots, C_{n+k}$. If this is indeed the case, then $p_{n,k} = \Pr(A_{n,k}) = (k+1)/(k+2)$. Below, we will formalize this.

For each $j = 1, 2, \ldots, n-1$, define the event

$$B_{n,k,j} = \text{"during the } j\text{-th iteration, person } P_1 \text{ chooses one of the chairs } C_1, C_n, C_{n+1}, \ldots, C_{n+k}\text{."}$$

(We consider the first two lines of the algorithm to be the first iteration.)

Since exactly one of these events is guaranteed to occur, the Law of Total Probability (Theorem **??**) implies that

$$\Pr\left(A_{n,k}\right) = \sum_{j=1}^{n-1} \Pr\left(A_{n,k} \mid B_{n,k,j}\right) \cdot \Pr\left(B_{n,k,j}\right).$$

Define the event

$$B_{n,k} = \text{“a uniformly random element from the set } \{1, n, n+1, \ldots, n+k\} \text{ is not equal to } n\text{.”}$$

Then for each j with $1 \leq j \leq n-1$, we have

$$\Pr\left(A_{n,k} \mid B_{n,k,j}\right) = \Pr\left(B_{n,k}\right) = \frac{k+1}{k+2}.$$

It follows that

$$\begin{aligned}
p_{n,k} &= \Pr\left(A_{n,k}\right) \\
&= \sum_{j=1}^{n-1} \frac{k+1}{k+2} \cdot \Pr\left(B_{n,k,j}\right) \\
&= \frac{k+1}{k+2} \sum_{j=1}^{n-1} \Pr\left(B_{n,k,j}\right) \\
&= \frac{k+1}{k+2},
\end{aligned}$$

because the latter summation is equal to 1.