COMP 2804 — Assignment 3

Due: March 21, before 4:30pm, in the course drop box in Herzberg 3115. Note that 3115 is open from 8:30am until 4:30pm.

Assignment Policy: Late assignments will not be accepted. Students are encouraged to collaborate on assignments, but at the level of discussion only. When writing the solutions, they should do so in their own words. Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.

Important note: When writing your solutions, you must follow the guidelines below.

- The answers should be concise, clear and neat.
- When presenting proofs, every step should be justified.
- Assignments should be stapled or placed in an unsealed envelope.

Substantial departures from the above guidelines will not be graded.

Question 1: On the first page of your assignment, write your name and student number.

Solution:
Name: Luke Skywalker
Student Number: 100001138

Question 2: You flip a fair coin four times. Define the four events (recall that zero is even)

$$\begin{aligned} A &= \text{"the coin comes up heads an odd number of times"}, \\ B &= \text{"the coin comes up heads an even number of times"}, \\ C &= \text{"the coin comes up tails an odd number of times"}, \\ D &= \text{"the coin comes up tails an even number of times"}. \end{aligned}$$

- Determine $\Pr(A)$, $\Pr(B)$, $\Pr(C)$, $\Pr(D)$, $\Pr(A \mid C)$, and $\Pr(A \mid D)$. Show your work.
- Are there any two events in the sequence A, B, C, and D that are independent? Justify your answer.

Solution: Out of four coins, the only numbers of heads that could be odd are 1 and 3.

Therefore,

$$\Pr(A) = \frac{\binom{4}{1} + \binom{4}{3}}{2^4}$$

$$= \frac{8}{16} = \frac{1}{2}$$

If the coin comes up heads an odd number of times, it does not come up an even number of times. If it comes up an even number of times, it does not come up an odd number of times.

$$\Pr(B) = 1 - \Pr(A) = \frac{1}{2}$$

The problem is identical if we replace all the heads with tails.

$$\Pr(C) = \Pr(D) = \frac{1}{2}$$

The total number of coin tosses is an even number (4) and even numbers are the sum of two odd numbers. Therefore, if the coin comes up tails an odd number of times, then the coin has to come up heads an odd number of times as well.

$$\Pr(A \mid C) = 1$$

By a similar logic, if the number of tails were even then the number of heads would have to be even as well to add up to 4 coin tosses.

$$\Pr(A \mid D) = 0$$

Since knowledge of any of the four events would provide us with information on whether all its other three events were occuring or not, all the four events are not conditionally independent.

Question 3: Answer the following questions:

1. Suppose a fair coin is tossed 10 times. What is the probability of obtaining exactly 4 Heads in 10 tosses?

2. Suppose a fair die is rolled 10 times. What is the probability of getting "6" exactly 4 times?

Solution: The probability of obtaining exactly four heads is equal to the sum of the individual probabilities of every sequence of coin tosses that generates 4 heads and 6 tails.

$$= \binom{10}{4} \Pr(Head)^4 \Pr(Tail)^6$$

$$= \binom{10}{4} \cdot \frac{1}{2^4} \cdot \frac{1}{2^6}$$

$$= \binom{10}{4} \cdot \frac{1}{1024} = \frac{105}{512}$$

The probability of obtaining exactly four "6"s is similar:

$$= \binom{10}{4} \Pr(6)^4 \Pr(\neg 6)^6$$
$$= \binom{10}{4} \cdot \frac{1}{6^4} \cdot \frac{5^6}{6^6} \approx 0.054$$

Question 4: In Section 5.4.1, we have seen the different cards that are part of a standard deck of 52 cards.

- You get a uniformly random hand of three cards from a standard deck of 52 cards. Determine the probability that this hand contains an Ace, a King, and a Queen. Show your work.
- You get a uniformly random hand of three cards from the 13 spades. Determine the probability that this hand contains an Ace, a King, and a Queen. Show your work.

Solution: Since this is a deck, each card drawn modifies the remaining distribution for further draws.

- For any ordering of the Ace, King or Queen, the probabilities of drawing each card will always be (4/52), (4/51) and (4/50) respectively.

  Therefore the probability of drawing an Ace, King and Queen from a deck of 52 cards would be:

  $$3! \cdot \frac{4}{52} \cdot \frac{4}{51} \cdot \frac{4}{50} = \frac{16}{5525}$$

- By similar reasoning, the probabilities will be (1/13), (1/12) and (1/11) respectively.

  The answer will be:

  $$3! \cdot \frac{1}{13} \cdot \frac{1}{12} \cdot \frac{1}{11} = \frac{1}{286}$$

Question 5: Suppose in the Minor Hockey League Championship play off series between Nepean Pirates and Kanata Thunders, there are three possible playoff games planned. A team that wins two games is declared the champion. Outcome of each game is either a win or a loss - there are no ties! The first game is played in the Pirates Arena, the Second game is played in the Thunder's arena, and if required the third game will be played in the Pirates arena. A team wins with a probability of 2/3rd in its home arena, and with a probability of 1/3rd in the opposition's arena. What is the probability that Nepean Pirates will be declared the Champion?

Solution: There are four possible permutations of victory and loss that would account for the Nepean Pirates being declared the Champion:

- The Nepean Pirates win the first two games and lose the third. This would have a probability of $(2/3) * (1/3) * (1/3) = (2/27)$
- The Nepean Pirates win the first and last game and lose the second. This would have a probability of $(2/3) * (2/3) * (2/3) = (8/27)$
- The Nepean Pirates win the last two games and lose the first. This would have a probability of $(1/3) * (1/3) * (2/3) = (2/27)$
- The Nepean Pirates win all three games. This would have a probability of $(2/3) * (1/3) * (2/3) = (4/27)$

By summing up the probabilities, we can see that the Nepean Pirates will win the Championship with a probability of (16/27)

Question 6: Suppose that 8% of all bicycle racers use steroids, that a bicyclists who uses steroids tests positive for steroids 96% of the time, and that a bicyclist who does not use steroids tests positive for steroids 9% of times. What is the probability that a randomly selected bicyclist who tests positive for steroids actually uses steroids?

Solution:

$$\Pr(Steroids \mid Positive) = \frac{\Pr(Positive \mid Steroids) \cdot \Pr(Steroids)}{\Pr(Positive)}$$

$$= \frac{0.96 \cdot 0.08}{\Pr(Positive \mid Steroids) \cdot \Pr(Steroids) + \Pr(Positive \mid \neg Steroids) \cdot \Pr(\neg Steroids)}$$

$$\approx 0.48$$

The probability that a randomly selected bicyclist who tests posittive for steroids actually uses steroids is 48.12%.

Question 7: There are three cards in a box. Both sides of Card 1 are Black. Both sides of Card 2 are Red. One side of Card 3 is Black and the other side is Red. We pick a card uniformly at random and observe only one of its side. Answer the following questions:
(a) If the side we observed is Black, what is the probability that the other side is also black?
(b) What is the probability that the opposite side is the same color as the side that we observed?

Solution: Let's name all of the sides.

- The two sides of Card 1 can be B1 and B2.

- The two sides of Card 2 can be R1 and R2.
- The two sides of Card 3 can be B3 and R3.

If we observe one of the sides B1, B2, B3, then two of those sides have a black side opposite them (B1 and B2). The last side has a red side opposite it (B3 has R3 on the opposite side). Since the probability of picking any given side is equal (it's a uniform distribution), the probability that the other side will be black if the side we observed is black is 2/3.

Out of all the six sides we can observe, the only sides with the same color on the opposite side are: B1, B2, R1, R2. Hence, the probability that the opposite side is the same color as the side we observed is 4/6 (which is just 2/3).

Question 8: Suppose a store has seveal boxes of candies, where each box is either of Type A or Type B. The stores inventory consists of 60% of Type A boxes and 40% of Type B boxes. Each Type A box contains 70% sweet candies and 30% sour candies. Each Type B box contains 30% sweet candies and 70% sour candies. These boxes are not labelled, and hence by looking at a box we cannot determine which type it is. A customer goes to the store and decides to buy a box. The storekeeper hands a random box to this customer. The customer opens the box, tastes a random candy, and then needs to make a decision, whether the box is likely to be of Type A or Type B.

1. Determine the probability that if the candy the customer tasted is sweet, what is the probability that: It is a Type A box? It is a Type B box?
2. Determine the probability that if the candy the customer tasted is sour, what is the probability that: It is a Type A box? It is a Type B box.

Solution:

1.

$$\begin{aligned}\Pr(A \mid sweet) &= \frac{\Pr(sweet \mid A) \cdot \Pr(A)}{\Pr(sweet \mid A) \cdot \Pr(A) + \Pr(sweet \mid B) \cdot \Pr(B)} \\ &= \frac{0.7 \cdot 0.6}{0.7 \cdot 0.6 + 0.4 \cdot 0.3} = \frac{7}{9}\end{aligned}$$

Therefore,

$$\Pr(B \mid sweet) = 1 - \frac{7}{9} = \frac{2}{9}$$

2.

$$\begin{aligned}\Pr(A \mid sour) &= \frac{\Pr(sour \mid A) \cdot \Pr(A)}{\Pr(sour \mid A) \cdot \Pr(A) + \Pr(sour \mid B) \cdot \Pr(B)} \\ &= \frac{0.3 \cdot 0.6}{0.3 \cdot 0.6 + 0.4 \cdot 0.7} = \frac{9}{23}\end{aligned}$$

Therefore,

$$\Pr(B \mid sour) = 1 - \frac{9}{23} = \frac{14}{23}$$

Question 9: Three factories A, B, and C, produce 25%, 35%, and 40%, of the total output of snow shovels in a season, respectively. Out of their production, 5%, 4%, and 2%, respectively, are defective shovels. A shovel is chosen uniformly at random and found to be defective. What is the probability that this shovel came from: a) Factory A? b) Factory B?, c) Factory C?

Solution:

$$\Pr(\text{Def}) = \Pr(\text{Def} \mid A) \cdot \Pr(A) + \Pr(\text{Def} \mid B) \cdot \Pr(B) + \Pr(\text{Def} \mid C) \cdot \Pr(C)$$

$$= 0.25 \cdot 0.05 + 0.35 \cdot 0.04 + 0.4 \cdot 0.02 = 0.0345$$

1.
$$\Pr(A \mid \text{Def}) = \frac{\Pr(\text{Def} \mid A) \cdot \Pr(A)}{\Pr(\text{Def})}$$
$$= \frac{0.25 \cdot 0.05}{0.0345} = \frac{25}{69}$$

2.
$$\Pr(B \mid \text{Def}) = \frac{\Pr(\text{Def} \mid B) \cdot \Pr(B)}{\Pr(\text{Def})}$$
$$= \frac{0.35 \cdot 0.04}{0.0345} = \frac{28}{69}$$

3.
$$\Pr(C \mid \text{Def}) = \frac{\Pr(\text{Def} \mid C) \cdot \Pr(C)}{\Pr(\text{Def})}$$
$$= \frac{0.4 \cdot 0.02}{0.0345} = \frac{16}{69}$$

Question 10: In a manufacturing process of an article, defects of one type occurs with probability 0.1 and the defects of other type occurs with probability 0.05. Assume that the two defects are independent of each other. What is the probability that

1. An article doesn't have both types of defects.
2. An article is defective.
3. An article has only one type of defect, given that it is defective.

Solution: We start with the following known values: $\Pr(\text{Def1}) = 0.1$, $\Pr(\text{Def2}) = 0.05$

1. Since the two defects are conditionally independent of each other, we can simplify this to:

$$\Pr(\text{Doesn't have both defects}) = 1 - \Pr(\text{Def1}) \cdot \Pr(\text{Def2})$$
$$= 1 - 0.1 \cdot 0.05 = 0.995$$

2.
$$\Pr(\text{Def1} \cup \text{Def2}) = \Pr(\text{Def1}) + \Pr(\text{Def2}) - \Pr(\text{Def1} \cap \text{Def2})$$

Since they're conditionally independent, we can simplify this to:

$$\Pr(\text{Def1} \cup \text{Def2}) = \Pr(\text{Def1}) + \Pr(\text{Def2}) - \Pr(\text{Def1}) \cdot \Pr(\text{Def2})$$

$$= 0.1 + 0.05 - 0.1 \cdot 0.05 = 0.145$$

3. We can rephrase this as the difference in probability between an article being defective and it having both defects.

$$\Pr(\text{Exactly one defect}) = \Pr(\text{Def1} \cup \text{Def2}) - \Pr(\text{Def1} \cap \text{Def2})$$

$$= 0.145 - 0.1 \cdot 0.05 = 0.14$$

Therefore, the probability that it has exactly one defect given that it is defective is:

$$\Pr(\text{Exactly one defect} \mid \text{Defective}) = \frac{\Pr(\text{Defective} \mid \text{Exactly one defect}) \cdot \Pr(\text{Exactly one defect})}{\Pr(\text{Defective})}$$

$$= \frac{1 \cdot 0.14}{0.145} = \frac{28}{29}$$

Question 11: (Bonus Problem:) An airplane has seats for $n + k$ passengers, where $n \geq 2$ and $k \geq 0$ are integers. In all n passengers are going to board the flight, and each of them has been issued a boarding pass with their seat number. Passengers board the airplane one by one. The first passenger, when he boards the flight, observes that his seat number is not printed very well on his boarding pass, and hence occupies a random seat. The second passenger takes his seat, if it is empty, otherwise chooses one of the random unoccupied seats. The third passenger takes her seat, if it is empty, otherwise chooses one of the random unoccupied seats. And this process is repeated till all the n-passengers have boarded the flight. We can assume that the seat numbers are readable for all passengers, except for the first one. What is the probability that the last person finds his/her seat occupied?

Solution: We can rephrase the problem in a manner that'll make it more intuitive to think about. When a passenger enters the airplane and finds their seat taken, rather than moving to find a new seat themselves they simply oust the passenger currently in their seat and make that passenger look for a new seat. The problem is still identical (you end up having a fungible passenger assigned to a new location when a collission occurs).

In this newly phrased problem, the first passenger occupies a random seat. The seat can be in any of the four following ranges:

- Her original seat. (In which case the problem ends with the last person not finding their seat occupied)
- The seat of the last occupant. (In which case the problem ends with the last person finding their seat occupied)
- A seat currently assigned to another occupant who isn't the last occupant. (In which case she'll be ousted eventually by that occupant)
- One of the k extra seats assigned to no-one. (In which case the problem ends with the last person not finding their seat occupied)

We can consequently ignore any of the $n-2$ assigned seats that don't belong to the first or last passenger. If the first passenger ever sits in any of them, she'll be ousted to move to a new seat eventually. After all the first $n-1$ passengers have entered the airplane, the first passenger will have to either be in her original seat, in the last passenger's seat, or in any of the k unassigned seats.

As a result, the probability that the last person finds their seat occupied is just $1/(k+2)$.