Online Matching

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Problem

Basic Problem

A problem motivated from which advertisement to display on the web

Manufacturer	Products	Ad Amount	Total Budget
I	A	$1	100
II	A,B	$ 2	100

Online Queries for Products A and B.

Question: Which Manufacturer's Ad should be shown given that we can display exactly one advertisement at a time?

Complication: We don't know how many queries, and with what distribution, for each product we will receive.

Example Query Sequences

Manufacturer	Products	Ad Amount	Total Budget
I	A	$1	100
II	A,B	$ 2	100

Sample Query Sequences:

1. 50 for B followed by 100 for A
2. 100 for A followed by 50 for B
3. intermix of 100 for A and 50 for B
4. intermix of ? for A and ? for B

In Cases 1-3, it is best to assign all B's to Manufacturer II and all A's to Manufacturer I, with a total revenue of $200.

What to do in Case 4?

Competitive Ratio

Online v/s Offline

Competitive Ratio

Ratio of the value returned by an Online Algorithm in comparison to the (best) Offline algorithm.

What is the largest value of $c \leq 1$, such that

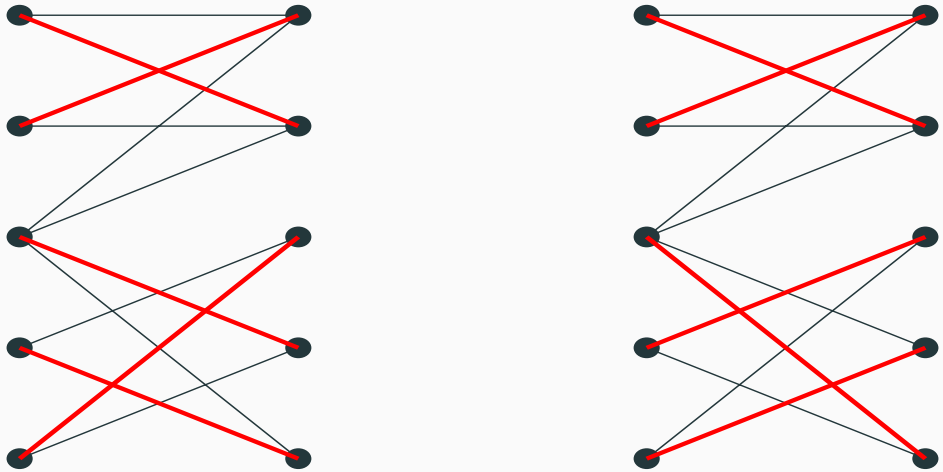
$$\frac{\textbf{Value (online algorithm)}}{\textbf{Value (off-line algorithm)}} \geq c$$

Bipartite Matching

Bipartite Matching

Let $G = (V = L \cup R, E)$ be a bipartite graph where the vertex set V consists of the sets L and R (referred to as 'left' and 'right' sets) and a set E of edges (v, w) where $v \in L$ and $w \in R$.

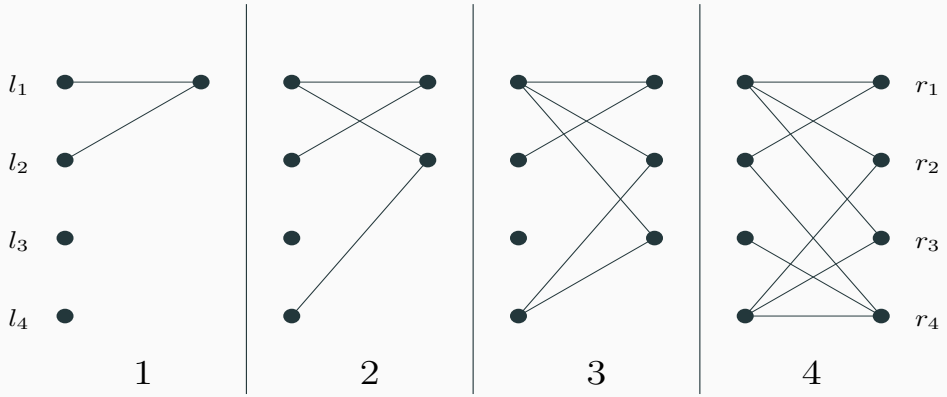
The set $M \subseteq E$ is a matching in G if no two edges in M share a vertex.



Online Matching Problem

Input: All the vertices in the set L are known in advance, but the vertices in R and the edges are presented over time. At each time instant $t \in \{1, 2, 3, \dots\}$, a new vertex $r_t \in R$ and all its incident edges arrive.

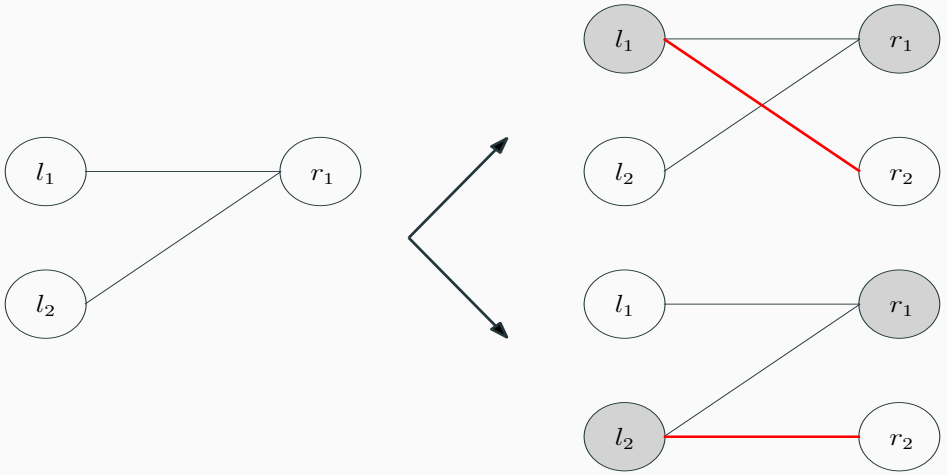
Task: The online matching algorithm needs to decide among all the currently unmatched neighbors of r_t in the set L to which vertex (if any) r_t should be matched. The vertex r_t remains matched to that vertex for the rest of the algorithm.



Output: Find a matching M of the largest possible size. Find M such that the ratio $\frac{|M|}{|M^*|}$ is as large as possible, where M^* is (offline) maximum matching in G.

Lower Bound on Deterministic Algorithms

Consider a bipartite graph on 4 vertices, where $L = \{l_1, l_2\}$ and $R = \{r_1, r_2\}$. At the first time step the algorithm is presented with the vertex r_1 and the two incident edges (r_1, l_1) and (r_1, l_2).



Adversary chooses what to do in the next time stamp and therefore the Competitive Ratio $= \frac{1}{2}$

Greedy Bipartite Matching Algorihm

Greedy Online Matching Algorithm:
At time step t:
Match r_t to any of the unmatched neighbors in the set L

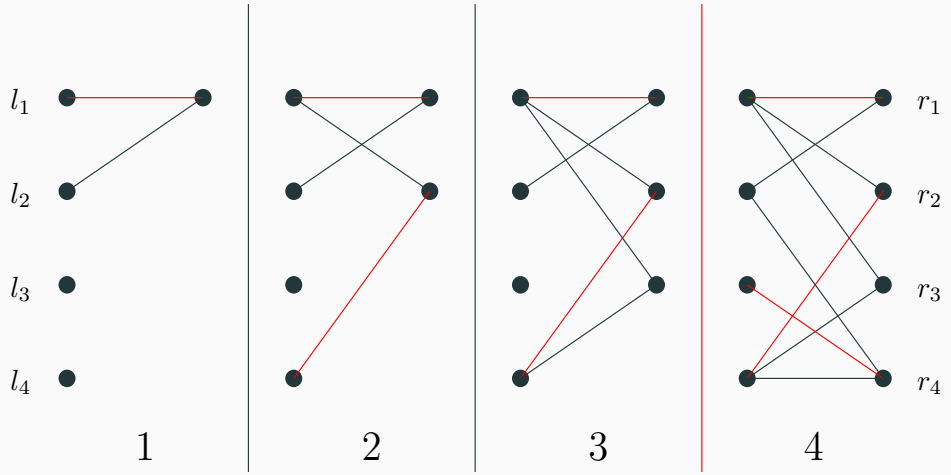


Figure 1: Competitive Ratio: $\frac{|M|}{|M*|} = \frac{3}{4}$

We will show that the Greedy Online Matching Algorithm has a competitive ratio $\geq \frac{1}{2}$

Competitive Ratio of Greedy Algorithm

Competitive Ratio

Online greedy matching algorithm is $\frac{1}{2}$-competitive.

Adwords

Advertising on the Web

Setting:

1. A set of bids from advertisers for search queries
2. Budget for each advertiser
3. CTR for each advertiser
4. Limit on the number of displayed ads

Objectives:

1. Ensure that advertiser has bid for the query
2. If ad is clicked, advertiser has budget to pay
3. Maximize the total revenue = value of the bid $\times$ CTR for the ad.

Uncertainty: Online Search queries

Advertising on the Web (contd.)

Simplifications:

1. Only display one advertisement per search query
2. Each advertiser has the same starting budget
3. All CTR's are the same
4. All bids are binary (either 0 or 1)

Greedy Algorithm

Greedy Ad Algorithm

On the arrival of a search query q:

Among the advertisers who have some remaining budget, pick any of them who has bid 1 for q

Advertiser	Search Query	Initial Budget
A_1	x	\$4
A_2	x, y	\$4

Competitive Ratio

GA for Ads

Greedy algorithm is $\frac{1}{2}$-competitive

Proof: Homework.

Idea: Look at the proof of the greedy online bipartite matching algorithm.

Balance Algorithm

Balance Algorithm

Balance Algorithm

Assign the query to the advertiser who bids for the query and has the highest remaining budget.

Break the ties deterministically.

Advertiser	Search Query	Initial Budget
A_1	x	\$4
A_2	x, y	\$4

Balance Algorithm (contd.)

2-Advertisers

The balance algorithm is $\frac{3}{4}$-competitive for 2-advertisers.

Many Advertisers $\rightarrow$ b-Matching

A bipartite graph $G = (L \cup R, E)$

Vertices in R come in an online manner along with the edges incident to them.

Parameter $b > 0$ is a fixed positive integer.

When a vertex $w \in R$ is revealed to the algorithm, possibly match it one of its neighbors $v \in L$ provided that the number of vertices matched to v so far by the algorithm is $< b$

Whatever decision that we make for w cannot be altered on the arrival of future vertices of R.

BALANCE Algorithm

BALANCE Algorithm

For each vertex $w \in R$ in order of its appearance:

Among all the neighbors of w in L that have been matched $< b$ times, match w to that neighbor (if any) that is matched to the fewest.

AdWords Problem

An alternate view of the problem:

1. Vertices in $L = \{1, 2, \ldots, N\}$ are advertisers, where each of them have a daily budget of \$1
2. Each advertiser bids a small amount $\epsilon > 0$ for a set of keywords of their liking.
3. The set R comprises of keyword queries that arrive in an online manner.
4. Each query keyword needs to be assigned to an advertiser (if any) who has bid for that keyword and has some remaining budget $\geq \epsilon$.
5. If the query is assigned to an advertiser, its budget is decreased by ϵ and we generate a revenue of ϵ.

AdWords Problem (contd.)

BALANCE algorithm assigns the query to the advertiser who has

1. Bid for that keyword
2. Has remaining budget $\geq \epsilon$
3. Among all those advertisers has the largest remaining budget.

Problem: Maximize the revenue generated by the algorithm, i.e., maximize the sum total of the budget spent by the advertisers.

An Example

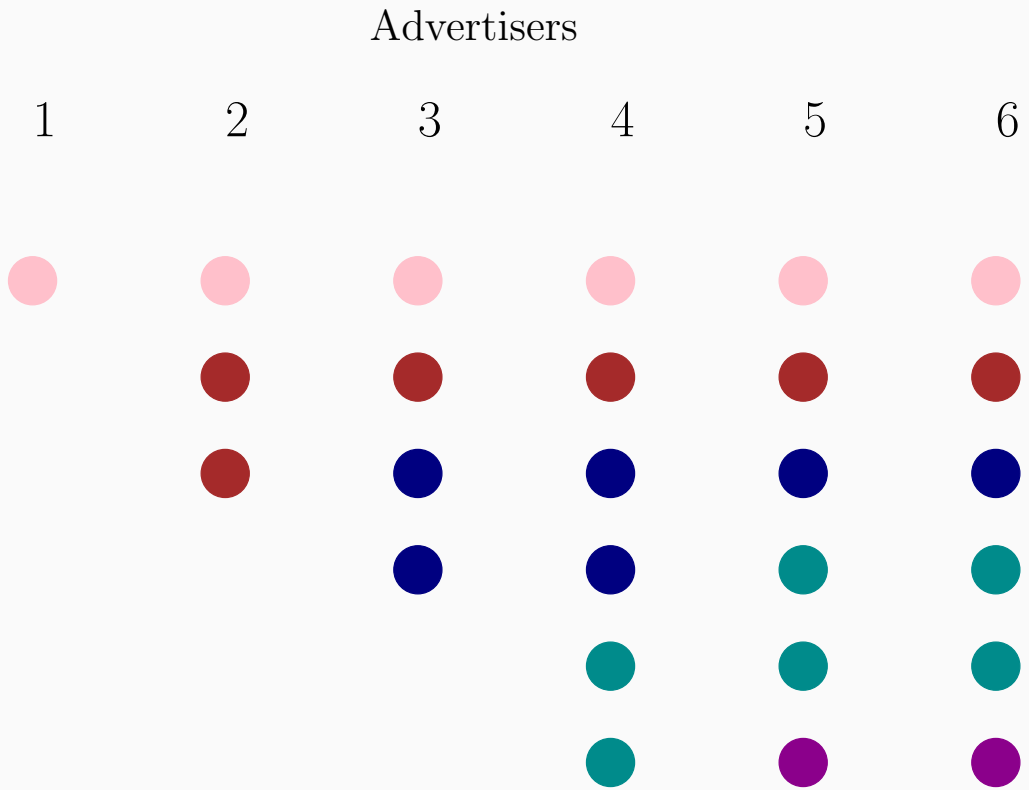


Figure 2: BALANCE with 6 advertisers numbered 1 to 6. Each has a budget of \$1 and can pay for 6 queries. Advertiser i bids for keywords $\{K_1, \ldots, K_i\}$. Thirty-six online queries arrive: first 6 for K_1 (pink dots), followed by next 6 for K_2 (dark red),... BALANCE handles 26 queries whereas optimal can handle all 36 queries.

Extending the Example

Setup:

- L has N vertices (advertisers) $1, \ldots, N$, each with a budget of \$1
- N keywords $K_1, \ldots, K_N$
- Advertiser i bids only for the keywords $\{K_1, \ldots, K_i\}$
- Set $\epsilon = \frac{1}{N}$
- Each advertiser can pay for at most N queries

Query Sequence:

- Total of N^2 queries
- First N queries are for the keyword K_1
- Next N queries are for the keyword K_2
- $\ldots$
- $\ldots$
- Last N queries are for the keyword K_N

Offline Revenue $= N$

Analysis of BALANCE

- First N queries corresponding to the keyword K_1 are distributed evenly among all the advertisers

- Next N queries corresponding to the keyword K_2 are distributed among the advertisers $2, \ldots, N$

- In general, N queries for the keyword K_i are distributed evenly among advertisers $i, \ldots, n$ provided that they have sufficient remaining budget

Which queries the advertiser N receives?

- at least one query of type K_1
- at least one query of type K_2
- ...
- at least $\lfloor \frac{N}{N-i} \rfloor$ queries of type K_i

Analysis of BALANCE (contd.)

When does N-th advertiser runs out of budget ?

$$\begin{aligned} N &\leq \left\lfloor \frac{N}{N} \right\rfloor + \left\lfloor \frac{N}{N-1} \right\rfloor + \cdots + \left\lfloor \frac{N}{N-i} \right\rfloor \\ &\leq N\left(\frac{1}{N} + \frac{1}{N-1} + \cdots + \frac{1}{N-i} \right) \end{aligned}$$

Condition on i

$i \approx N(1 - \frac{1}{e})$

Recall that n-th Harmonic Number $H_n = \sum_{i=1}^{n} \frac{1}{i} \approx \ln n$.

Express $\frac{1}{N} + \frac{1}{N-1} + \cdots + \frac{1}{N-i}$ as the difference of two Harmonic numbers.

Analysis of BALANCE (contd.)

Competitive Ratio

BALANCE algorithm's competitive ratio is at most $1 - \frac{1}{e}$

Proof: The above example illustrates that the revenue of Balance is $N(1 - \frac{1}{e})$.

Offline, we will assign first N queries to Advertiser 1, next N queries to Advertiser 2, and so on.

Total offline revenue $= N$.

Thus, Competitive Ratio = $\frac{N(1-\frac{1}{e})}{N} = 1 - \frac{1}{e}$.

□

Non-uniform bids

Suppose 2-advertisers bid for the same set of queries but they have different bid amounts:

Advertiser	Bid Amount	Budget
A_1	\$1	\$110
A_2	\$10	\$100

Consider a set of 10 queries:

Revenue(offline) $= 100$

Revenue(Balance Algorithm) $= 10$

Non-uniform bids (contd.)

New Idea: For query q:

Suppose advertiser A_i bids amount x_i for q:

Let f_i be the **fraction** of the unspent budget of A_i

Define $\Psi_i = x_i(1 - e^{-f_i})$

Assign q to the advertiser that has the largest value of Ψ_i

Non-uniform bids - Example

Advertiser	Bid Amount	Budget
A_1	$x_1 = \$1$	\$110
A_2	$x_2 = \$10$	\$100

$\Psi_i = x_i(1 - e^{-f_i})$

Advertiser	1st Query	2nd Query	3rd Query
Ψ_{A_1}	$1(1 - e^{-1}) = 0.63$	$1(1 - e^{-1}) = 0.63$	$1(1 - e^{-1}) = 0.63$
Ψ_{A_2}	$10(1 - e^{-1}) = 6.3$	$10(1 - e^{-.9}) = 5.9$	$10(1 - e^{-.8}) = 5.5$

Conclusions

1. The competitive ratio of Balance is $1 - \frac{1}{e}$
2. Non-uniform bid algorithm has a competitive ratio of $1 - \frac{1}{e}$
3. No online algorithm can have a competitive ratio greater than $1 - \frac{1}{e}$
4. Main Reference: Aranyak Mehta, Amin Saberi, Umesh V. Vazirani, Vijay V. Vazirani: AdWords and generalized online matching. Journal of ACM 54(5): 22 (2007)
5. Bala Kalyanasundaram, Kirk Pruhs: An optimal deterministic algorithm for online b-matching. Theoretical Computer Science 233(1-2): 319-325 (2000)