Quick Review of Probability

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Outline

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Sample Space & Events

Basic Definition

Definitions

Sample Space S = Set of Outcomes.

Events $\mathcal{E}$ = Subsets of S.

Probability is a function from subsets $A \subseteq S$ to positive real numbers between $[0, 1]$ such that:

1. $Pr(S) = 1$
2. For all $A, B \subseteq S$ if $A \cap B = \emptyset$, $Pr(A \cup B) = Pr(A) + Pr(B)$.
3. If $A \subset B \subseteq S$, $Pr(A) \leq Pr(B)$.
4. Probability of complement of A, $Pr(\bar{A}) = 1 - Pr(A)$.

Basic Definition

Examples:

1. Flipping a fair coin:
   $S = \{H, T\}$;
   $\mathcal{E} = \{\emptyset, \{H\}, \{T\}, S = \{H, T\}\}$
2. Flipping fair coin twice:
   $S = \{HH, HT, TH, TT\}$;
   $\mathcal{E} = \{\emptyset, \{HH\}, \{HT\}, \{TH\}, \{TT\},$
   $\{HH, TT\}, \{HH, TH\}, \{HH, HT\},$
   $\{HT, TH\}, \{HT, TT\}, \{TH, TT\},$
   $\{HH, HT, TH\}, \{HH, HT, TT\}, \{HH, TH, TT\},$
   $\{HT, TH, TT\}, S = \{HH, HT, TH, TT\}\}$
3. Rolling fair die twice:
   $S = \{(i, j) : 1 \leq i, j \leq 6\}$;
   $\mathcal{E} = \{\emptyset, \{1, 1\}, \{1, 2\}, \ldots, S\}$

Random Variable

Expectation

Definition

A random variable X is a function from sample space S to Real numbers, $X : S \to \Re$.
Expected value of a discrete random variable X is given by $E[X] = \sum_{s \in S} X(s) * Pr(X = X(s))$.

Note: Its a misnomer to say X is a random variable, it's a function.

Example: Flip a fair coin and define the random variable $X : \{H, T\} \to \Re$ as

$$X = \begin{cases} 1 & \text{Outcome is Heads} \\ 0 & \text{Outcome is Tails} \end{cases}$$

$E[X] = \sum_{s \in \{H,T\}} X(s) * Pr(X = X(s)) = 1 * \frac{1}{2} + 0 * \frac{1}{2} = \frac{1}{2}$

Linearity of Expectation

Definition

Consider two random variables X, Y such that $X, Y : S \to \Re$, then $E[X+Y] = E[X] + E[Y]$.
In general, consider n random variables $X_1, X_2, \dots, X_n$ such that $X_i : S \to \Re$, then $E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$.

Example: Flip a fair coin n times and define n random variable $X_1, \dots, X_n$ as

$$X_i = \begin{cases} 1 & \text{Outcome is Heads} \\ 0 & \text{Outcome is Tails} \end{cases}$$

$E[X_1 + \cdots + X_n] = E[X_1] + \cdots + E[X_n] = \frac{1}{2} + \cdots + \frac{1}{2} = \frac{n}{2}$
$=$ Expected # of Heads in n tosses.

Proof of Linearity of Expectation

$$
\begin{aligned}
E[X+Y] &= \sum_{\omega \in S} (X+Y)[\omega] \cdot P(\omega) \\
&= \sum_{\omega \in S} (X[\omega] + Y[\omega]) \cdot P(\omega) \\
&= \sum_{\omega \in S} (X[\omega] \cdot P(\omega) + Y[\omega] \cdot P(\omega)) \\
&= \sum_{\omega \in S} X[\omega] \cdot P(\omega) + \sum_{\omega \in S} Y[\omega] \cdot P(\omega) \\
&= E[X] + E[Y]
\end{aligned}
$$

This generalizes to the sum of n random variables:
$E[X_1 + \cdots + X_n] = E[X_1] + \cdots + E[X_n]$.

More detailed proof

$$
\begin{aligned}
E[X+Y] &= \sum_x \sum_y (x+y)P(X=x,Y=y) \\
&= \sum_x \sum_y xP(X=x,Y=y) + \sum_x \sum_y yP(X=x,Y=y) \\
&= \sum_x \sum_y xP(X=x,Y=y) + \sum_y \sum_x yP(X=x,Y=y) \\
&= \sum_x x \sum_y P(X=x,Y=y) + \sum_y y \sum_x P(X=x,Y=y) \\
&= \sum_x xP(X=x) + \sum_y yP(Y=y) \\
&= E[X]+E[Y]
\end{aligned}
$$

Geometric Distribution

Geometric Distribuition

Definition

Perform a sequence of independent trials till the first success. Each trial succeeds with probability p (and fails with probability $1-p$).
A Geometric Random Variable X with parameter p is defined to be equal to $n \in N$ if the first $n-1$ trials are failures and the n-th trial is success. Probability distribution function of X is
$Pr(X=n) = (1-p)^{n-1}p$.

Let Z to be the $r.v.$ that equals the # failures before the first success, i.e. $Z = X - 1$.

Problem: Evaluate $E[X]$ and $E[Z]$.

To show: $E[Z] = \frac{1-p}{p}$ and $E[X] = 1 + \frac{1-p}{p} = \frac{1}{p}$.

Computation of $E[Z]$

$Z =$ # failures before the first success.

Set $q = 1 - p$.

- $Pr(Z = k) = q^k p$
- $\frac{1}{1-q} = \sum_{k=0}^{\infty} q^k$ (for $0 < q < 1$)
- $\frac{1}{(1-q)^2} = \sum_{k=0}^{\infty} kq^{k-1}$

$$\begin{aligned} E[Z] &= \sum_{k=0}^{\infty} kPr(Z = k) \\ &= \sum_{k=0}^{\infty} kq^k p \\ &= pq \sum_{k=0}^{\infty} kq^{k-1} \\ &= \frac{pq}{(1-q)^2} = \frac{1-p}{p} \end{aligned}$$

Examples

Examples:

1. Flipping a fair coin till we get a Head:
   $p = \frac{1}{2}$ and $E[X] = \frac{1}{p} = 2$
2. Roll a die till we see a 6:
   $p = \frac{1}{6}$ and $E[X] = \frac{1}{p} = 6$
3. Keep buying LottoMax tickets till we win (assuming we have 1 in 33294800 chance).
   $p = \frac{1}{33294800}$ and $E[X] = \frac{1}{p} = 33,294,800$.

Coupon Collector Problem

Coupon's Collector Problem

Problem Definition

There are a total of n different types of coupons (Pokemon cards). A cereal manufacturer has ensured that each cereal box contains a coupon. Probability that a box contains any particular type of coupon is $\frac{1}{n}$. What is the expected number of boxes we need to buy to collect all the n coupons?

Define r.v. $N_1, N_2, \ldots, N_n$, where $N_i =$# of boxes bought till the i-th coupon is collected.
Each N_i is a geometric random variable.

Coupon's Collector Problem Contd.

Let $N = \sum_{j=1}^{n} N_i$; Note $N_1 = 1$

$E[N_j] = \frac{1}{\text{Pr of success in finding the } j^{th} \text{ coupon}} = \frac{1}{\frac{n-j+1}{n}}$

$E[N] = \sum_{j=1}^{n} \frac{n}{n-j+1} = nH_n$, where $H_n = n$-th Harmonic Number.

$H_n = \sum_{i=1}^{n} \frac{1}{i}$ and $\ln n \leq H_n \leq \ln n + 1$.

Thus, $E[N] = nH_n \approx n \ln n$,

Is $E[N] = nH_n = n \ln n$ a good estimate?

What is the probability that $E[N]$ exceeds $2nH_n$? Applying Markov's Inequality: $Pr(X > s) \leq \frac{E[X]}{s}$ $Pr(N > 2nH_n) < \frac{E[N]}{2nH_n} = \frac{nH_n}{2nH_n} = \frac{1}{2}$

Can we have a better bound?

Next: We show $Pr(N > n \ln n + cn) < \frac{1}{e^c}$

Pr. of missing a coupon after $n \ln n + cn$ boxes have been bought $= (1 - \frac{1}{n})^{n \ln n + nc} \leq e^{-\frac{1}{n}(n \ln n + cn)} = \frac{1}{ne^c}$.

Pr. of missing at least one coupon $\leq n(\frac{1}{ne^c}) = \frac{1}{e^c}$.

References

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