

Estimating Frequency Moments F_0 and F_2

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Frequency Moments

Definition

Let $A = (a_1, a_2, \dots, a_n)$ be a stream, where elements are from universe $U = \{1, \dots, u\}$. Let $m_i = \#$ of elements in A that are equal to i . The k -th frequency moment $F_k = \sum_{i=1}^u m_i^k$, where $0^0 = 1$.

Example: $F_k = \sum_{i=1}^u m_i^k$

$A = (3, 2, 4, 7, 2, 2, 3, 2, 2, 1, 4, 2, 2, 2, 1, 1, 2, 3, 2)$ and $m_1 = m_3 = 3$, $m_2 = 10$,
 $m_4 = 2$, $m_7 = 1$, $m_5 = m_6 = 0$

$$F_0 = \sum_{i=1}^7 m_i^0 = 3^0 + 10^0 + 3^0 + 2^0 + 0^0 + 0^0 + 1^0 = 5$$

(# of Distinct Elements in A)

$$F_1 = \sum_{i=1}^7 m_i^1 = 3^1 + 10^1 + 3^1 + 2^1 + 0^1 + 0^1 + 1^1 = 19$$

(# of Elements in A)

$$F_2 = \sum_{i=1}^7 m_i^2 = 3^2 + 10^2 + 3^2 + 2^2 + 0^2 + 0^2 + 1^2 = 123$$

(Surprise Number)

...

Streaming Problem

Find frequency moments in a stream

Input: A stream A consisting of n elements from universe $U = \{1, \dots, u\}$.

Output: Estimate Frequency Moments F_k 's for different values of k .

Our Task: Estimate F_0 and F_2 using sublinear space

Reference: The space complexity of estimating frequency moments by Noga Alon, Yossi Matias, and Mario Szegedy, Journal of Computer Systems and Science, 1999.

Estimating F_0

Computation of F_0

Input: Stream $A = (a_1, a_2, \dots, a_n)$, where each $a_i \in U = \{1, \dots, u\}$.

Output: An estimate $\hat{F}_0$ of number of distinct elements F_0 in A such that $Pr\left(\frac{1}{c} \leq \frac{\hat{F}_0}{F_0} \leq c\right) \geq 1 - \frac{2}{c}$ for some constant c using sublinear space.

Algorithm

Algorithm for Estimating F_0

Input: Stream A and a hash function $h : U \rightarrow U$

Output: Estimate $\hat{F}_0$

Step 1: Initialize $R := 0$

Step 2: For each elements $a_i \in A$ do:

1. Compute binary representation of $h(a_i)$
2. Let r be the location of the rightmost 1 in the binary representation
3. if $r > R$, $R := r$

Step 3: Return $\hat{F}_0 = 2^R$

Space Requirements = $O(\log u)$ bits

Correctness

Observation 1

Let d to be smallest integer such that $2^d \geq u$ (d -bits are sufficient to represent numbers in U)

Observation 1

$$Pr(\text{rightmost 1 in } h(a_i) \text{ is at location } \geq r + 1) = \frac{1}{2^r}$$

Proof: For that to happen the last r bits in $h(a_i)$ must be 0. Since h is a hash function from universal family of hash functions, this happens with probability $(\frac{1}{2})^r$.

□

Observation 2

For $a_i \neq a_j$, $Pr(\text{rightmost 1 in } h(a_i) \geq r + 1 \text{ and rightmost 1 in } h(a_j) \geq r + 1) = \frac{1}{2^{2r}}$

Proof: $h(a_i)$ and $h(a_j)$ are independent as $a_i \neq a_j$.

$$\begin{aligned} Pr(\text{rightmost 1 in } h(a_i) \geq r + 1 \text{ and rightmost 1 in } h(a_j) \geq r + 1) &= Pr(\text{rightmost 1 in } h(a_i) \geq r + 1) \times Pr(\text{rightmost 1 in } h(a_j) \geq r + 1) \\ &= \frac{1}{2^r} \times \frac{1}{2^r} = \frac{1}{2^{2r}} \end{aligned}$$

□

Observations 3

Fix $r \in \{1, \dots, d\}$. $\forall x \in A$, define indicator r.v:

$$I_x^r = \begin{cases} 1, & \text{if the rightmost 1 is at location } \geq r + 1 \text{ in } h(x) \\ 0, & \text{otherwise} \end{cases}$$

Let $Z^r = \sum I_x^r$ (sum is over **distinct** elements of A)

Observation 3

The following holds:

1. $E[I_x^r] = \frac{1}{2^r}$
2. $Var[I_x^r] = \frac{1}{2^r} \left(1 - \frac{1}{2^r}\right)$
3. $E[Z^r] = \frac{F_0}{2^r}$
4. $Var[Z^r] \leq E[Z^r]$

Observation 3.1

$$E[I_x^r] = \frac{1}{2^r}$$

Proof: $E[I_x^r] = 1 \times \Pr(I_x^r = 1) + 0 \times \Pr(I_x^r = 0) = \frac{1}{2^r}$

Note that $\Pr(I_x^r = 1)$ corresponds to

$\Pr(\text{rightmost 1 in } h(x) \text{ is at location } \geq r + 1) = \frac{1}{2^r}$ by Observation 1.



Observation 3.2

$$\text{Var}[I_x^r] = E[I_x^{r2}] - E[I_x^r]^2 = \frac{1}{2^r} \left(1 - \frac{1}{2^r}\right)$$

Proof: Note that the variance of a random variable X is given by

$$\text{Var}[X] = E[(X - E[X])^2] = E[X^2] - E[X]^2.$$

$$E[I_x^{r2}] = 1^2 \text{Pr}(I_x^r = 1) = \frac{1}{2^r}$$

$$\text{Then } E[I_x^{r2}] - E[I_x^r]^2 = \frac{1}{2^r} - \left(\frac{1}{2^r}\right)^2 = \frac{1}{2^r} \left(1 - \frac{1}{2^r}\right)$$

□

Observation 3.3

$$E[Z^r] = \frac{F_0}{2^r}$$

Proof: Let $A' \subseteq A$ be the set of distinct elements of A .

Note that $F_0 = |A'|$.

By definition $Z^r = \sum_{x \in A'} I_x^r$

$$\text{Then, } E[Z^r] = E\left[\sum_{x \in A'} I_x^r\right] = \sum_{x \in A'} E[I_x^r] = \sum_{x \in A'} \frac{1}{2^r} = \frac{F_0}{2^r}$$

□

Observation 3.4

$$\text{Var}[Z^r] = \frac{F_0}{2^r} \left(1 - \frac{1}{2^r}\right) \leq \frac{F_0}{2^r} = E[Z^r]$$

Proof: $\text{Var}[Z^r] = \text{Var}\left[\sum_{x \in A'} I_x^r\right]$

For two independent random variables X and Y ,

$$\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y].$$

$$\text{Var}[Z^r] = \text{Var}\left[\sum_{x \in A'} I_x^r\right] = \sum_{x \in A'} \text{Var}[I_x^r] = F_0 \frac{1}{2^r} \left(1 - \frac{1}{2^r}\right) \leq \frac{F_0}{2^r} = E[Z^r]$$

□

Observation 4

If $2^r > cF_0$, $Pr(Z^r > 0) < \frac{1}{c}$

Proof: Recall Markov's inequality for a random variable X ,
 $Pr(X \geq s) \leq \frac{E[X]}{s}$, where $s > 0$ and X takes positive values.

What is the number of distinct elements $x \in A$, whose hash map $h(x)$ has its rightmost 1 in position $\geq r + 1$?

$$= Z^r = \sum_{x \in A'} I_x^r$$

What is $Pr(Z^r > 0)$? $\Leftrightarrow$ What is $Pr(Z^r \geq 1)$.

By Markov's inequality: $Pr(Z^r \geq 1) \leq \frac{E[Z^r]}{1} = E[Z^r] = \frac{F_0}{2^r} < \frac{1}{c}$.

□

Chebyshev's Inequality

$$Pr(|X - E[X]| \geq \alpha) \leq \frac{Var[X]}{\alpha^2}$$

Proof: Recall Markov's inequality for a random variable X , $Pr(X \geq s) \leq \frac{E[X]}{s}$, where $s > 0$ and X takes positive values.

Now

$$\begin{aligned} Pr(|X - E[X]| \geq \alpha) &= Pr((X - E[X])^2 \geq \alpha^2) \\ &\leq \frac{E[(X - E[X])^2]}{\alpha^2} \\ &= \frac{Var[X]}{\alpha^2} \end{aligned}$$

□

Observation 5

If $c2^r < F_0$, $Pr(Z^r = 0) < \frac{1}{c}$

Proof: Recall Chebyshev's inequality $Pr(|X - E[X]| \geq \alpha) \leq \frac{Var[X]}{\alpha^2}$.

For a random variable X , $Pr(X = 0) \leq Pr(|X - E[X]| \geq E[X])$, as the event $|X - E[X]| \geq E[X]$ includes $X \leq 0$ and $X \geq 2E[X]$.

Now, $Pr(Z^r = 0) \leq Pr(|Z^r - E[Z^r]| \geq E[Z^r])$.

$$\begin{aligned} Pr(Z^r = 0) &\leq Pr(|Z^r - E[Z^r]| \geq E[Z^r]) \\ &\leq \frac{Var[Z^r]}{E[Z^r]^2} \\ &\leq \frac{E[Z^r]}{E[Z^r]^2} \\ &= \frac{1}{E[Z^r]} = \frac{2^r}{F_0} < \frac{1}{c} \end{aligned}$$

□

Claim

Set $\hat{F}_0 = 2^R$. We have $Pr\left(\frac{1}{c} \leq \frac{\hat{F}_0}{F_0} \leq c\right) \geq 1 - \frac{2}{c}$

Proof We have that

Observation 4: if $2^r > cF_0$, $Pr(Z^r > 0) < \frac{1}{c}$

Observation 5, if $c2^r < F_0$, $Pr(Z^r = 0) < \frac{1}{c}$

When do we produce a wrong answer?

Case 1: $\hat{F}_0 = 2^R > cF_0$, but this happens with $Pr(Z^R > 0) < \frac{1}{c}$

Case 2: $c2^R = c\hat{F}_0 < F_0$, but this happens with $Pr(Z^R = 0) < \frac{1}{c}$

Therefore, with probability $\leq \frac{2}{c}$, we produce a wrong answer.

$\implies$ with probability $\geq 1 - \frac{2}{c}$, we produce the right answer, i.e.,

$$Pr\left(\frac{1}{c} \leq \frac{\hat{F}_0}{F_0} \leq c\right) \geq 1 - \frac{2}{c}$$

□

Further Improvements

Execute the algorithm s times in parallel
(with independent hash functions)
Let R to the median value among these runs
Return $\hat{F}_0 = 2^R$

Note: Algorithm uses $O(s \log u)$ bits.

Claim

For $c > 4$, there exists $s = O(\log \frac{1}{\epsilon})$, $\epsilon > 0$, such that
 $Pr(\frac{1}{c} \leq \frac{\hat{F}_0}{F_0} \leq c) \geq 1 - \epsilon.$

Technique: Median + Chernoff Bounds

Improving success probability (contd.)

i -th Run of the Algorithm:

Step 1: Initialize $R_i := 0$

Step 2: For each elements $a_i \in A$ do:

1. Compute binary representation of $h(a_i)$
2. Let r be the location of the rightmost 1 in the binary representation
3. if $r > R_i$, $R_i := r$

Step 3: Return R_i

Let $R = \text{Median}(R_1, R_2, \dots, R_s)$

Define $X_1, \dots, X_s$ be indicator random variables:

$$X_i = \begin{cases} 0, & \text{if success, i.e. } \frac{1}{c} \leq \frac{2^{R_i}}{F_0} \leq c \\ 1, & \text{otherwise} \end{cases}$$

1. $E[X_i] = \Pr(X_i = 1) \leq \frac{2}{c} = \beta < \frac{1}{2}$ (Since $c > 4$)
2. Let $X = \sum_{i=1}^s X_i$ = Number of failures in s runs
3. $E[X] \leq s\beta < \frac{s}{2}$
4. If $X < \frac{s}{2}$, then $\frac{1}{c} \leq \frac{2^R}{F_0} \leq c$
($R = \text{Median}(R_1, R_2, \dots, R_s)$)

Chernoff Bounds

If r.v. X is sum of independent identical indicator r.v. and $0 < \delta < 1$,

$$Pr(X \geq (1 + \delta)E[X]) \leq e^{-\frac{\delta^2 E[X]}{3}}$$

Proof: See my notes

An example: Toss a fair coin n -times. Let X be the total number of heads obtained in these n -trials. Evaluate $Pr(X \geq \frac{3}{4}n)$

$$\begin{aligned} Pr(X \geq \frac{3}{4}n) &= Pr(X \geq (1 + \frac{1}{2})\frac{n}{2}) \\ &= Pr(X \geq (1 + \frac{1}{2})E[X]) \\ &\leq e^{-\frac{(\frac{1}{2})^2 E[X]}{3}} \\ &= e^{-\frac{n}{24}} \end{aligned}$$

Claim

For any $\epsilon > 0$, if $s = O(\log \frac{1}{\epsilon})$, $Pr(X < \frac{s}{2}) \geq 1 - \epsilon$

Proof: We show that $Pr(X \geq \frac{s}{2}) < \epsilon$.

$$E[X] = s\beta < \frac{s}{2}$$

$$\begin{aligned} Pr(X \geq \frac{s}{2}) &= Pr(X - E[X] \geq \frac{s}{2} - E[X]) \\ &= Pr(X - E[X] \geq \frac{s}{2} - s\beta) \\ &= Pr(X - E[X] \geq \frac{\frac{1}{2} - \beta}{\beta} s\beta) \\ &= Pr(X - E[X] \geq \frac{\frac{1}{2} - \beta}{\beta} E[X]) \\ &= Pr(X \geq \left(1 + \frac{\frac{1}{2} - \beta}{\beta}\right) E[X]) \end{aligned}$$

$$\begin{aligned} \Pr(X \geq \frac{s}{2}) &= \Pr(X \geq \left(1 + \frac{\frac{1}{2} - \beta}{\beta}\right) E[X]) \\ &\leq e^{-\frac{1}{3} \left(\frac{\frac{1}{2} - \beta}{\beta}\right)^2 E[X]} \end{aligned}$$

We want $e^{-\frac{1}{3} \left(\frac{\frac{1}{2} - \beta}{\beta}\right)^2 E[X]} \leq \epsilon$

Substitute $E[X] = s\beta$ and we have

$$-\frac{1}{3} \left(\frac{\frac{1}{2} - \beta}{\beta}\right)^2 s\beta \leq \ln \epsilon$$

$$\Leftrightarrow s \geq \frac{3}{\beta} \left(\frac{\beta}{\frac{1}{2} - \beta}\right)^2 \ln \frac{1}{\epsilon}$$

$$\implies \text{if } s \in O(\ln \frac{1}{\epsilon}), \Pr(X \geq \frac{s}{2}) < \epsilon.$$

□

Estimating F_2

Estimating F_2

Input: Stream A and hash function $h : U \rightarrow \{-1, +1\}$

Output: Estimate $\hat{F}_2$ of $F_2 = \sum_{i=1}^u m_i^2$

Algorithm (Tug of War)

Step 1: Initialize $Y := 0$.

Step 2: For each element $x \in U$, evaluate $r_x = h(x)$.

Step 3: For each element $a_i \in A$, $Y := Y + r_{a_i}$

Step 4: Return $\hat{F}_2 = Y^2$

Correctness

Observation 1

Observation 1

$$E[r_i] = 0$$

Proof: $E[r_i] = -1 \times \frac{1}{2} + 1 \times \frac{1}{2} = 0$



Observation 2

$$\text{Let } Y = \sum_{i=1}^u r_i m_i$$

$$E[Y^2] = \sum_{i=1}^u m_i^2 = F_2$$

Proof:

$$\begin{aligned} E[Y^2] &= E\left[\sum_{i=1}^u r_i m_i \sum_{j=1}^u r_j m_j\right] \\ &= E\left[\sum_{i=1}^u r_i^2 m_i^2 + \sum_{i,j:i \neq j} r_i r_j m_i m_j\right] \\ &= \sum_{i=1}^u E[r_i^2 m_i^2] + \sum_{i,j:i \neq j} E[r_i r_j m_i m_j] \\ &= \sum_{i=1}^u m_i^2 E[r_i^2] + \sum_{i,j:i \neq j} m_i m_j E[r_i] E[r_j] \\ &= \sum_{i=1}^u m_i^2 = F_2 \end{aligned}$$

Observation 3

$Pr(|Y^2 - E[Y^2]| \geq \sqrt{2}cE[Y^2]) \leq \frac{1}{c^2}$ for any positive constant c . (I.e., Y^2 approximates $F_2 = E[Y^2]$ within a constant factor with $Pr \geq 1 - \frac{1}{c^2}$)

Proof: Recall Chebyshev's inequality $Pr(|X - E[X]| \geq \alpha) \leq \frac{Var[X]}{\alpha^2}$.

Now, $Pr(|Y^2 - E[Y^2]| \geq \sqrt{2}cE[Y^2]) \leq \frac{Var[Y^2]}{(\sqrt{2}cE[Y^2])^2}$.

$$Var[Y^2] = E[Y^4] - E[Y^2]^2$$

$$\begin{aligned} E[Y^4] &= E \left[\sum_{i=1}^u r_i m_i \sum_{j=1}^u r_j m_j \sum_{k=1}^u r_k m_k \sum_{l=1}^u r_l m_l \right] \\ &= \sum_{i=1}^u E[r_i^4 m_i^4] + 6 \sum_{1 \leq i < j \leq u} E[r_i^2 r_j^2 m_i^2 m_j^2] \\ &= \sum_{i=1}^u m_i^4 + 6 \sum_{1 \leq i < j \leq u} m_i^2 m_j^2 \end{aligned}$$

Observation 4 contd.

$$\begin{aligned} \text{Var}[Y^2] &= E[Y^4] - E[Y^2]^2 \\ &= \sum_{i=1}^u m_i^4 + 6 \sum_{1 \leq i < j \leq u} m_i^2 m_j^2 - \left(\sum_{i=1}^u m_i^2 \right)^2 \\ &= 4 \sum_{1 \leq i < j \leq u} m_i^2 m_j^2 \\ &\leq 2F_2^2 \end{aligned}$$

Now, $\frac{\text{Var}[Y^2]}{(\sqrt{2}cE[Y^2])^2} = \frac{2F_2^2}{(\sqrt{2}cE[Y^2])^2} = \frac{2F_2^2}{2c^2F_2^2} = \frac{1}{c^2}$

Thus, $Pr(|Y^2 - E[Y^2]| \geq \sqrt{2}cE[Y^2]) \leq \frac{\text{Var}[Y^2]}{(\sqrt{2}cE[Y^2])^2} = \frac{1}{c^2}$

□

Improving Variance

Improving the Variance

Execute the algorithm k times (using independent hash functions) resulting in $Y_1^2, Y_2^2, \dots, Y_k^2$.

$$\text{Output } \bar{Y}^2 = \frac{1}{k} \sum_{i=1}^k Y_i^2$$

Observations:

1. $E[\bar{Y}^2] = E[Y^2] = F_2$
2. $Var[\bar{Y}^2] = \frac{1}{k} Var[Y^2]$
(Note: $Var[cX] = c^2 Var[X]$)
3. $Pr(|\bar{Y}^2 - E[\bar{Y}^2]| \geq \sqrt{\frac{2}{k}} c E[\bar{Y}^2]) \leq \frac{1}{c^2}$
4. Set $k = O(\frac{1}{\epsilon^2})$, we have
 $Pr(|\bar{Y}^2 - E[\bar{Y}^2]| \geq \epsilon c E[\bar{Y}^2]) \leq \frac{1}{c^2}$

Complexity

Algorithm (Tug of War)

Step 1: Initialize $Y := 0$.

Step 2: For each element $x \in U$, evaluate $r_x = h(x)$.

Step 3: For each element $a_i \in A$, $Y := Y + r_{a_i}$

Step 4: Return $\hat{F}_2 = Y^2$

- Need to store Y and $(r_1, r_2, \dots, r_u)$.
 Y requires $O(\log n)$ bits.
- Total space required is $O(\log n + \log u)$.

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3. Notes on Algorithm Design by A.M
4. Several Lecture Notes (Tim Roughgarden, Ankush Moitra, Lap Chi Lau, Yufei Tao, John Augustine,...)