# Introduction to Matrices

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# Introduction

# Matrices

1. A Rectangular Array
2. Operations: Addition; Multiplication; Diagonalization; Transpose; Inverse; Determinant
3. Row Operations; Linear Equations; Gaussian Elimination
4. Types: Identity; Symmetric; Diagonal; Upper/Lower Traingular; Orthogonal; Orthonormal
5. Transformations - Eigenvalues and Eigenvectors
6. Rank; Column and Row Space; Null Space
7. Applications: Page Rank, Dimensionality Reduction, Recommender Systems, . . .

# Utility Matrix $M$

A Matrix $M$ where rows represent users, columns items, and entries in $M$ represents the ratings.

$$M = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 2 & 0 & 4 & 4 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 1 & 0 & 2 & 2 \end{bmatrix} =$$

$$\begin{bmatrix} .13 & -.02 & .01 \\ .41 & -.07 & .03 \\ .55 & -.1 & .04 \\ .68 & -.11 & .05 \\ .15 & .59 & -.65 \\ .07 & .73 & .67 \\ .07 & .29 & -.32 \end{bmatrix} \begin{bmatrix} 12.5 & 0 & 0 \\ 0 & 9.5 & 0 \\ 0 & 0 & 1.35 \end{bmatrix} \begin{bmatrix} .56 & .59 & .56 & .09 & .09 \\ -.12 & .02 & -.12 & .69 & .69 \\ .40 & -.8 & .40 & .09 & .09 \end{bmatrix}$$

Questions: How to guess missing entries? How to guess ratings for a new user? ...

# Matrix Vector Product

Matrix-vector product: $Ax = b$

$$\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \times 4 + 1 \times -2 \\ 3 \times 4 + 4 \times -2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

# Matrix Vector Product

$Ax = b$ as linear combination of columns:

$$\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = 4 \begin{bmatrix} 2 \\ 3 \end{bmatrix} -2 \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

# Matrix-Matrix Product

- Matrix-matrix product $A = BC$:

$$\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 6 & 16 \end{bmatrix}$$

- $A = BC$ as sum of rank $1$ matrices:

$$\begin{aligned} \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 4 \end{bmatrix} &= \begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 8 \\ 6 & 12 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 8 \\ 6 & 16 \end{bmatrix} \end{aligned}$$

# Row Reduced Echelon Form

Let $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 8 \\ 10 & 16 & 24 \end{bmatrix}$

1st Pivot: Replace $r_2$ by $r_2 - r_1$, and $r_3$ by $r_3 - 5r_1$:

$$\begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 8 \\ 0 & 6 & 24 \end{bmatrix}$$

2nd Pivot: Replace $r_3$ by $r_3 - 3r_2$:

$$\begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

# RREF contd.

Divide the first row by $2$, the second row by $2$:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Replace $r_1$ by $r_1 - r_2$:

$$R = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

# Rank

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 8 \\ 10 & 16 & 24 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} = R$$

Definitions:

- Rank $=$ Number of non-zero pivots $= 2$
- Basis vectors of row space = rows corresponding to non-zero pivots in $R$
  $v_1 = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$
- Basis vectors of column space = Columns of $A$ corresponding to non-zero pivots of $R$.

  $u_1 = \begin{bmatrix} 2 \\ 2 \\ 10 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 2 \\ 4 \\ 16 \end{bmatrix}$
- $A$ as sum of the product of rank $1$ matrices

$$A = u_1 v_1^T + u_2 v_2^T = \begin{bmatrix} 2 \\ 2 \\ 10 \end{bmatrix} \begin{bmatrix} 1 & 0 & -4 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 16 \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 \end{bmatrix}$$

# Null Space

Null space of $A$ = All vectors $x$ such that $Ax = 0$.
This includes the $0$ vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Is there a vector $x = (x_1, x_2, x_3) \in R^3$, such that
$Ax = x_1 \begin{bmatrix} 2 \\ 2 \\ 10 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \\ 16 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 8 \\ 24 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x = (1, -1, 1/4)$, or any of its scalar multiples, satisfies $Ax = 0$

Dimension of Null Space of $A$= Number of columns $(A)$ - rank$(A)$= $3 - 2 = 1$

## Spaces for $A$

Let $A$ be $m \times n$ matrix with real entries.
Let $R$ be RREF of $A$ consisting of $r \leq \min\{m, n\}$ non-zero pivots.

1. $rank(A) = r$
2. Column space is a subspace of $R^m$ of dimension $r$, and its basis vectors are the columns of $A$ corresponding to the non-zero pivots in $R$.
3. Row space is a subspace of $R^n$ of dimension $r$, and its basis vectors are the rows of $R$ corresponding to the non-zero pivots.
4. The null-space of $A$ consists of all the vectors $x \in R^n$ satisfying $Ax = 0$. They form a subspace of dimension $n - r$.

# Eigenvalues

# Eigenvalues and Eigenvectors

Given an $n \times n$ matrix $A$.
A non-zero vector $v$ is an eigenvector of $A$, if $Av = \lambda v$ for some scalar $\lambda$.
$\lambda$ is the eigenvalue corresponding to vector $v$.

**Example**

Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

Observe that

$$\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Thus, $\lambda_1 = 5$ and $\lambda_2 = 1$ are the eigenvalues of $A$.
Corresponding eigenvectors are $v_1 = [1, 3]$ and $v_2 = [1, -1]$, as $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$.

# Computation of Eigenvalues and Eigenvectors

Given an $n \times n$ matrix $A$, we want to find eigenvalues $\lambda$'s and the corresponding eigenvectors that satisfy $Av = \lambda v$.

We can express $Av = \lambda v$ as $(A - \lambda I)v = 0$, where $I$ is $n \times n$ identity matrix.

Suppose $B = A - \lambda I$.

If $B$ is invertible, than the only solution of $Bv = 0$ is $v = 0$.
Reason: $v = B^{-1}Bv = B^{-1}(Bv) = B^{-1}0 = 0$.

$\implies B$ isn't invertible and hence the determinant of $B$ is $0$.

We solve the equation $det(A - \lambda I) = 0$ to obtain eigenvalues $\lambda$.

Once we know an eigenvalue $\lambda_i$, we can solve $Av_i = \lambda_i v_i$ to obtain the corresponding eigenvector $v_i$.

# Computation of Eigenvalues and Eigenvectors

Let us find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

$$det(A - \lambda I) = \begin{bmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)(4-\lambda) - 3 = 0$$

$\lambda^2 - 6\lambda + 5 = 0$, and the two roots are $\lambda_1 = 5$ and $\lambda_2 = 1$.

To find the eigenvector $v_1 = [a, b]$, we can solve $\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 5 \begin{bmatrix} a \\ b \end{bmatrix}$.

This gives: $2a + b = 5a$ and $b = 3a$. Thus $v_1 = [1, 3]$ is an eigenvector corresponding to $\lambda_1 = 5$.

Similarly, for $v_2$, we have $\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 1 \begin{bmatrix} a \\ b \end{bmatrix}$.

This gives $2a + b = a$, or $a = -b$. Thus, $v_2 = [1, -1]$ is an eigenvector corresponding to $\lambda_2 = 1$.

# Matrices with distinct eigenvalues

**Propertry**

Let $A$ be an $n \times n$ real matrix with $n$ distinct eigenvalues.
The corresponding eigenvectors are linearly independent.

**Proof:** Proof by contradiction. Let $\lambda_1, \ldots, \lambda_n$ be the distinct eigenvalues and $v_1, \ldots, v_n$ the corresponding eigenvectors, that are linearly dependent.

Assume $v_1, \ldots, v_{n-1}$ are L.I. (otherwise work with a smaller set).

Dependence $\implies \alpha_1 v_1 + \ldots + \alpha_{n-1} v_{n-1} + \alpha_n v_n = 0$, where $\alpha_n \neq 0$.

$\implies v_n = \frac{-\alpha_1}{\alpha_n} v_1 + \ldots + \frac{-\alpha_{n-1}}{\alpha_n} v_{n-1}$

Multiply by $A$: $A v_n = \lambda_n v_n = \frac{-\alpha_1}{\alpha_n} \lambda_1 v_1 + \ldots + \frac{-\alpha_{n-1}}{\alpha_n} \lambda_{n-1} v_{n-1}$

Multiply by $\lambda_n$: $\lambda_n v_n = \frac{-\alpha_1}{\alpha_n} \lambda_n v_1 + \ldots + \frac{-\alpha_{n-1}}{\alpha_n} \lambda_n v_{n-1}$

Subtract last two equations:
$0 = \frac{-\alpha_1}{\alpha_n} (\lambda_n - \lambda_1) v_1 + \ldots + \frac{-\alpha_{n-1}}{\alpha_n} (\lambda_n - \lambda_{n-1}) v_{n-1}$

Since, $\lambda_n - \lambda_i \neq 0, \implies$ the vectors $v_1, \ldots, v_{n-1}$ are linearly dependent.
A contradiction.

$\square$

# Matrices with distinct eigenvalues

Let $A$ be an $n \times n$ real matrix with $n$ distinct eigenvalues.
Let $\lambda_1, \ldots, \lambda_n$ be the distinct eigenvalues and let $x_1, \ldots, x_n$ be the corresponding eigenvectors, respectively. Let each $x_i = [x_{i1}, x_{i2}, \ldots, x_{in}]$.

Define an eigenvector matrix $X = \begin{bmatrix} x_{11} & x_{21} & \ldots & x_{n1} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1n} & x_{2n} & \ldots & x_{nn} \end{bmatrix}$

Define a diagonal $n \times n$ matrix $\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & \lambda_n \end{bmatrix}$

Consider the matrix product $AX$,

$$AX = A \begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \ldots & \lambda_n x_n \end{bmatrix} = X\Lambda$$

# Matrices with distinct eigenvalues

Since eigenvectors are linearly independent, we know that $X^{-1}$ exists.

Multiply by $X^{-1}$ on both the sides from left in $AX = X\Lambda$ and we obtain

$$X^{-1}AX = X^{-1}X\Lambda = \Lambda \tag{1}$$

and when we multiply on the right we obtain

$$AXX^{-1} = A = X\Lambda X^{-1} \tag{2}$$

**An Application of Diagonalization** $A = X\Lambda X^{-1}$

Consider $A^2 = (X\Lambda X^{-1})(X\Lambda X^{-1}) = X\Lambda(X^{-1}X)\Lambda X^{-1} = X\Lambda^2 X^{-1}$
$\implies A^2$ has the same set of eigenvectors as $A$, but eigenvalues are squared.

Similarly, $A^k = X\Lambda^k X^{-1}$.
Eigenvectors of $A^k$ are same as that of $A$ and its eigenvalues are raised to the power of $k$.

# Eigenvalues of $A^k$

Let $Av_i = \lambda_i v_i$

Consider: $A^2 v_i = A(Av_i) = A(\lambda_i v_i) = \lambda_i(Av_i) = \lambda_i(\lambda_i v_i) = \lambda_i^2 v_i$
$\implies A^2 v_i = \lambda_i^2 v_i$

**Eigenvalues of $A^k$**

For an integer $k > 0$, $A^k$ has the same eigenvectors as $A$, but the eigenvalues are $\lambda^k$.

# Symmetric Matrices

## Example

Consider symmetric matrix $S = \left[\begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix}\right]$.
Its eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = 2$ and the corresponding eigenvectors are $q_1 = (1/\sqrt{2}, 1/\sqrt{2})$ and $q_2 = (1/\sqrt{2}, -1/\sqrt{2})$, respectively.
Note that eigenvalues are real and the eigenvectors are orthonormal.

$$S = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

## Eigenvalues of Symmetric Matrices

All the eigenvalues of a real symmetric matrix $S$ are real. Moreover, all components of the eigenvectors of a real symmetric matrix $S$ are real.

# Symmetric Matrices (contd.)

**Property**

Any pair of eigenvectors of a real symmetric matrix $S$ corresponding to two different eigenvalues are orthogonal.

**Proof:** Let $q_1$ and $q_2$ be eigenvectors corresponding to $\lambda_1 \neq \lambda_2$, respectively.
We have $Sq_1 = \lambda_1 q_1$ and $Sq_2 = \lambda_2 q_2$.
Now $(Sq_1)^T = q_1^T S^T = q_1^T S = \lambda_1 q_1^T$, as $S$ is symmetric,
Multiply by $q_2$ on the right and we obtain $\lambda_1 q_1^T q_2 = q_1^T S q_2 = q_1^T \lambda_2 q_2$.
Since $\lambda_1 \neq \lambda_2$ and $\lambda_1 q_1^T q_2 = q_1^T \lambda_2 q_2$, this implies that $q_1^T q_2 = 0$ and thus the eigenvectors $q_1$ and $q_2$ are orthogonal.

□

**Symmetric matrices with distinct eigenvalues**

Let $S$ be a $n \times n$ symmetric matrix with $n$ distinct eigenvalues and let $q_1, \ldots, q_n$ be the corresponding orthonormal eigenvectors. Let $Q$ be the $n \times n$ matrix consiting of $q_1, \ldots, q_n$ as its columns. Then $S = Q\Lambda Q^{-1} = Q\Lambda Q^T$. Furthermore, $S = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \cdots + \lambda_n q_n q_n^T$

$$S = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = 4 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} + 2 \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

# Summary for Symmetric Matrices

**Theorem**

*For a real symmetric $n \times n$ matrix $S$, we have*

1. *All eigenvalues of $S$ are real.*
2. *$S$ can be expressed as $S = Q\Lambda Q^T$, where $Q$ consists of orthonormal basis of $R^n$ formed by $n$ eigenvectors of $S$, and $\Lambda$ is a diagonal matrix consisting of $n$ eigenvalues of $S$.*
3. *$S$ can be expressed as the sum of the product of rank $1$ matrices:*

$$S = \lambda_1 q_1 q_1^T + \ldots + \lambda_n q_n q_n^T$$

**Note:** Since $Q$ is a basis of $R^n$, any vector $x$ can be expressed as a linear combination $x = \alpha_1 q_1 + \ldots + \alpha_n q_n$

Consider $x \cdot q_i = (\alpha_1 q_1 + \ldots + \alpha_n q_n) \cdot q_i = \alpha_i$

# Inverse of Symmetric Matrices

**Claim**

$S = Q\Lambda Q^T$ and $S^{-1} = \frac{1}{\lambda_1} q_1 q_1^T + \ldots + \frac{1}{\lambda_n} q_n q_n^T$

Proof Sketch: $S = Q\Lambda Q^T = \lambda_1 q_1 q_1^T + \ldots + \lambda_n q_n q_n^T$

$SS^{-1} = (\lambda_1 q_1 q_1^T + \ldots + \lambda_n q_n q_n^T)(\frac{1}{\lambda_1} q_1 q_1^T + \ldots + \frac{1}{\lambda_n} q_n q_n^T) = I$ as $q_1, \ldots, q_n$ are orthonormal.

□

### Positive Definite Matrices

A symmetric matrix $S$ is positive definite if all its eigenvalues $> 0$.
It is positive semi-definite if all the eigenvalues are $\geq 0$.

### An Alternate Characterization

Let $S$ be a $n \times n$ real symmetric matrix. For all non-zero vectors $x \in R^n$, if $x^T S x > 0$ holds, then all the eigenvalues of $S$ are $> 0$.

**Proof:** Let $\lambda_i$ be an eigenvalue of $S$.
Let the corresponding unit eigenvector is $q_i$.
Note that $q_i^T q_i = 1$.
Since $S$ is symmetric, we know that $\lambda_i$ is real.
Now we have, $\lambda_i = \lambda_i q_i^T q_i = q_i^T \lambda_i q_i = q_i^T S q_i$.
But $q_i^T S q_i > 0$, hence $\lambda_i > 0$.

□

# Eigenvalue Identities

**Trace**

Let $\lambda_1, \ldots, \lambda_n$ be eigenvalues of $n \times n$ real matrix $A$.

trace$(A) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i$

**Determinant**

det$(A) = \prod_{i=1}^{n} \lambda_i$