

The Ski-Rental Problem

Deterministic and randomized competitive analysis



Model and objective

- Renting skis costs \$1 per day; buying costs \$ B .
- The total number t of skiing days is unknown to the online algorithm.
- Every day, the algorithm must choose whether to keep renting or to buy.

The offline optimum knows the season length.

Thus, $C(\text{OPT}) = \min\{t, B\}$.

Deterministic Strategy

1. Rent on days $1, 2, \dots, B - 1$.
 2. If the season continues, buy on day B .
- The algorithm pays for $B - 1$ rental days before committing to the purchase.

Note: A purchase on day d costs $d - 1 + B$ in total.

Deterministic upper bound

Let t be the number of skiing days.

Case 1: $t < B$. The algorithm only rents.

$$C(\text{ALG}) = t = C(\text{OPT}), \text{ Competitive Ratio } \text{CR} = \frac{C(\text{ALG})}{C(\text{OPT})} = 1.$$

Case 2: $t \geq B$. It buys on day B .

$$C(\text{ALG}) = (B - 1) + B = 2B - 1,$$

$$\text{CR} = \frac{2B - 1}{B} = \boxed{2 - \frac{1}{B}}.$$

Therefore the strategy is $(2 - 1/B)$ -competitive.

Matching deterministic lower bound

Deterministic Lower Bound

No deterministic online algorithm has competitive ratio smaller than $2 - 1/B$.

Adversarial construction

- Consider an arbitrary deterministic algorithm.
- Continue the season until the algorithm buys, or until day $T = 2B$.
- Let ℓ denote the number of days for which the algorithm rents.

There are two possibilities: $\ell = T$ or $\ell < T$.

Costs forced by the adversary

Case 1: $\ell = T = 2B$. The algorithm never buys.

$$C(\text{ALG}) = 2B, \quad C(\text{OPT}) = B, \quad \text{CR} = 2.$$

Case 2: $\ell < T$. The algorithm buys on day $\ell + 1$.

The adversary ends the season on that day, giving

$$C(\text{ALG}) = \ell + B, \quad C(\text{OPT}) = \min\{\ell + 1, B\}.$$

Thus,

$$\text{CR} = \frac{\ell + B}{\min\{\ell + 1, B\}}$$

Lower bound when $\ell + 1 \leq B$

- In this range, $C(\text{OPT}) = \ell + 1$.

- We need to show

$$\frac{\ell + B}{\ell + 1} \geq 2 - \frac{1}{B}.$$

- Because $B > 1$, this is equivalent to

$$(B - 1)^2 \geq \ell(B - 1).$$

- It holds since $\ell + 1 \leq B$ implies $\ell \leq B - 1$.

Thus, $\text{CR} \geq 2 - \frac{1}{B}$.

Lower bound when $\ell + 1 \geq B$

Here $C(\text{OPT}) = B$, and $\ell \geq B - 1$.

$$\text{CR} = \frac{\ell + B}{B} = 1 + \frac{\ell}{B} \geq 1 + \frac{B - 1}{B} = 2 - \frac{1}{B}.$$

- Together with the previous case, this proves the lower bound. - Optimal deterministic competitive ratio $2 - \frac{1}{B}$

A randomized improvement

- Assume that B is divisible by 4, avoiding rounding terms.
- Flip a fair coin and choose the buying day:
 - with probability $1/2$, buy on day $3B/4$;
 - with probability $1/2$, buy on day B .
- Let k be the (adversary-fixed) number of skiing days.
- We will analyze the expected cost in three ranges of k .

Case 1: $k < 3B/4$

Case 2: $k \geq B$

Case 3: $3B/4 \leq k < B$.

Randomized Case 1: $k < 3B/4$

- Both possible buying days are later than the final skiing day.
- Consequently, the algorithm never buys and

$$C(\text{ALG}) = k = C(\text{OPT}).$$

$$CR = 1$$

Randomized Case 2: $k \geq B$

- The offline optimum buys immediately: $C(\text{OPT}) = B$.

$$\begin{aligned}\mathbb{E}[C(\text{ALG})] &= \frac{1}{2} \left(\frac{3B}{4} - 1 + B \right) + \frac{1}{2}(2B - 1) \\ &= \frac{15B}{8} - 1.\end{aligned}$$

Therefore,

$$\text{CR} = \frac{\mathbb{E}[C(\text{ALG})]}{C(\text{OPT})} = \boxed{\frac{15}{8} - \frac{1}{B}} < \frac{15}{8}.$$

Randomized Case 3: $3B/4 \leq k < B$

- In this range, $C(\text{OPT}) = k$.
 - If the algorithm's buying day is $3B/4$, it rents $3B/4 - 1$ days and then buys.
 - If the buying day is B , the algorithm only rents, at cost k .
- We have $\mathbb{E}[C(\text{ALG})] = \frac{1}{2}k + \frac{1}{2} \left(\frac{3B}{4} - 1 + B \right)$
- Since $k \geq 3B/4$, we have $B \leq \frac{4k}{3}$
- Thus (dropping the negative -1 term)

$$\begin{aligned}\mathbb{E}[C(\text{ALG})] &\leq \frac{1}{2}k + \frac{1}{2} \left(k + \frac{4k}{3} \right) \\ &= \frac{5}{3}k.\end{aligned}$$

- Since $C(\text{OPT}) = k$, we obtain $\text{CR} \leq \frac{5}{3}$.

Summary so far

Case 1: $k < 3B/4$, $CR = 1$

Case 2: $k \geq B$, $CR = \frac{15}{8} - \frac{1}{B} \approx \frac{15}{8}$.

Case 3: $3B/4 \leq k < B$, $CR = \frac{5}{3}$.

Hence, $CR = \max\{1, \frac{15}{8}, \frac{5}{3}\} = \frac{15}{8} = 1.875$.

A variable first buying day

What is so special about choosing $3B/4$ as the first day of buying?

Replace $3B/4$ by cB , for $0 < c < 1$.

Assume cB is an integer for now.

For $k \geq B$:

$$X(c) = \frac{\frac{1}{2}((1+c)B - 1) + \frac{1}{2}(2B - 1)}{B} = \frac{3+c}{2} - \frac{1}{B}.$$

For $cB \leq k < B$:

$$Y(c, k) = \frac{\frac{1}{2}k + \frac{1}{2}((1+c)B - 1)}{k} = \frac{1}{2} + \frac{(1+c)B - 1}{2k}.$$

Worst case in the middle range

- For $cB \leq k < B$, the ratio is

$$Y(c, k) = \frac{1}{2} + \frac{(1+c)B - 1}{2k}.$$

- Since $(1+c)B - 1 > 0$, this decreases as k increases.
- Its maximum is at $k = cB$:

$$Y_{\max}(c) = Y(c, cB) = 1 + \frac{B - 1}{2cB}.$$

- For $k < cB$, the algorithm only rents and the ratio is 1.

Note: If cB is not integral, rounding the buying day affects only lower-order terms.

The optimal choice of c

- $X(c)$ increases with c , while $Y_{\max}(c)$ decreases.
- The best continuous choice balances them:

$$\frac{3+c}{2} - \frac{1}{B} = 1 + \frac{B-1}{2cB}.$$

- Simplifying gives

$$Bc^2 + (B-2)c - (B-1) = 0,$$

- Positive root is

$$c_B^* = \frac{2 - B + \sqrt{5B^2 - 8B + 4}}{2B}.$$

- For large B : $c_B^* \longrightarrow \frac{\sqrt{5}-1}{2} \approx 0.618$.

Comparison

- With the best c , the asymptotic ratio becomes

$$\lim_{B \rightarrow \infty} \min_c \max\{X(c), Y_{\max}(c)\} = \boxed{\frac{5 + \sqrt{5}}{4} \approx 1.809}.$$

- For $c = 3/4$: overall bound $15/8 \approx 1.875$.
- Optimized c : improved overall bound 1.809.

- A more complex randomized strategy (see notes) achieves the optimal

asymptotic ratio $\boxed{\frac{e}{e-1} \approx 1.582}$