Maximum Weight Independent Set

Anil Maheshwari

anil@scs.carleton.ca
School of Computer Science
Carleton University
Canada

Outline

Problem Statement

MWIS in Graphs

Input: An undirected graph $G = (V, E)$ where each vertex has a positive weight $w : V \to \Re^+$.

Output: A subset $S \subseteq V$ such that
(a) Independent: No two vertices in S are connected by an edge
(b) Maximality: Among all such independent sets, S has the maximum total weight, where $wt(S) = \sum_{s \in S} w(s)$.

Complexity Results on MWIS Problem

NP-Hardness Results:

- Decision version of MWIS problem is NP-Hard, both for unweighted and weighted graphs
- **NP**-Hard for cubic-graphs
- **NP**-Hard to approximate within a factor of $n^{1-\epsilon}$, for any $0 < \epsilon < 1$, [Hastad 2001]
- Can be solved in linear time for trees, bounded tree-width graphs, ...

A Greedy Randomized Algorithm

Greedy Randomized Algorithm

Consider the following straightforward greedy algorithm for approximating MWIS of an undirected weighted graph $G = (V, E)$.

Input: Graph $G = (V, E)$ on n vertices with $w : V \to \Re^+$.
Output: A set S that approximates the MWIS.

Step 1: Compute an ordering of vertices in V by using a uniform at random permutation. WLOG, let the ordering be $(v_1, \ldots, v_n)$.

Step 2: $S \leftarrow \emptyset$

Step 3: For each vertex v_i in order do
If none of its neighbors are in S, $S \leftarrow S \cup \{v_i\}$

Step 4: Return S

An Illustration

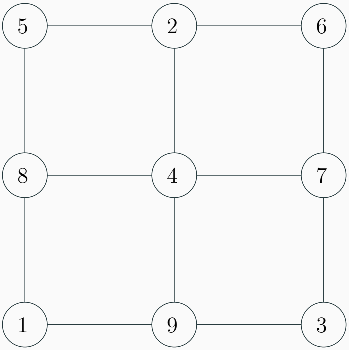

Figure 1: $S = \{1, 2, 3\}$

Observations on Greedy Algorithm

Observation 1

The set of vertices in S forms an independent set of G.

Observation 2

The algorithm is oblivious to weights of vertices.

Observation 3

The algorithm runs in $O(|V| + |E|)$ time.

Observations on Greedy Algorithm (contd.)

Observation 4

Let $v \in V$ be an arbitrary vertex of G and let its degree be $deg(v)$. Then

$$Pr(v \in S) \geq \frac{1}{deg(v) + 1}$$

where probability is over the random orderings of vertices in V.

Proof: Vertex v is placed in S if none of v's neighbors come before v in the ordering.

This occurs with probability $= \frac{1}{deg(v)+1}$

Moreover, it is possible that a neighbor w of v comes before v in the ordering, but it wasn't placed in S as one of w's neighbor (other than v) was in S.

Thus, $Pr(v \in S) \geq \frac{1}{deg(v)+1}$

□

Observations on Greedy Algorithm (contd.)

Observation 5

$E\left[\sum_{v \in S} w(v)\right] \geq \sum_{v \in V} \frac{w(v)}{deg(v)+1}$

Proof: Set up indicator random variable X_v for each vertex v, where

$$X_v = \begin{cases} 1, & \text{if } v \in S \\ 0, & \text{otherwise} \end{cases}$$

Note that $E[X_v] = Pr(X_v = 1) = Pr(v \in S) \geq \frac{1}{deg(v)+1}$

Now

$$\begin{aligned} E\left[\sum_{v \in S} w(v)\right] &= E\left[\sum_{v \in V} X_v w(v)\right] \\ &= \sum_{v \in V} E\left[X_v w(v)\right] = \sum_{v \in V} w(v) E\left[X_v\right] \\ &\geq \sum_{v \in V} \frac{w(v)}{deg(v)+1} \end{aligned}$$

□

Remarks on Observation 5

Remark 1

If max degree of any vertex in G is $\leq \Delta$, $E\left[\sum_{v \in S} w(v)\right] \geq \frac{1}{\Delta+1} \sum_{v \in V} w(v)$

Remark 2

Let I be any independent set of G. Then
$E\left[\sum_{v \in S} w(v)\right] \geq \sum_{v \in V} \frac{w(v)}{deg(v)+1} \geq \sum_{v \in I} \frac{w(v)}{deg(v)+1}$

Remark 3

Let I^* be a max weight independent set of G. Then
$E\left[\sum_{v \in S} w(v)\right] \geq \sum_{v \in I^*} \frac{w(v)}{deg(v)+1}$

Improvements

Recap

Step 1: Compute an ordering of vertices in V by using a uniform at random permutation. WLOG, let the ordering be $(v_1, \ldots, v_n)$.

Step 2: $S \leftarrow \emptyset$

Step 3: For each vertex v_i in order do
If none of the neighbors of v_i are in S, $S \leftarrow S \cup \{v_i\}$

Step 4: Return S

Remark 3

Let I^* be a max weight independent set of G. Then
$E\left[\sum_{v \in S} w(v)\right] \geq 1 \cdot \sum_{v \in I^*} \frac{w(v)}{deg(v)+1}$

The value 1 is called the *recoverable value* and we will see a method of Feige and Reichman [2014] to get a better value.

Upper Bound on Recoverable Value

Max Recoverable Value

The maximum value of r in the expression $E\left[\sum_{v \in S} w(v)\right] \geq r \cdot \sum_{v \in I} \frac{w(v)}{deg(v)+1}$ should be strictly less than 4 (unless **P=NP**).

Proof: Note that for the cubic graphs (i.e. graphs where each vertex has degree 3), the MWIS problem is **NP**-Hard. This also holds for unweighted cubic graphs.

If $r = 4$ in $E\left[\sum_{v \in S} w(v)\right] \geq r \cdot \sum_{v \in I*} \frac{w(v)}{deg(v)+1}$, then we have that

$$E\left[\sum_{v \in S} w(v)\right] \geq r \cdot \sum_{v \in I*} \frac{w(v)}{4} = \sum_{v \in I*} w(v).$$

Thus we may obtain an optimal MWIS in polynomial time for cubic graphs.

This is only feasible if **P=NP**.

□

FR14 Algorithm

Input: Graph $G = (V, E)$ on n vertices with $w : V \to \Re^+$.
Output: A set S that approximates the MWIS.

Step 1: Compute an ordering of vertices in V by using a uniform at random permutation. WLOG, let the ordering be $(v_1, \ldots, v_n)$.

Step 2: $F \leftarrow \emptyset$

Step 3: For each vertex v_i in order do
If at most one of the neighbors of v_i has been seen so far,
$F \leftarrow F \cup \{v_i\}$

Step 4: Compute a MWIS S of the induced graph on F.

Step 5: Return S

An Illustration

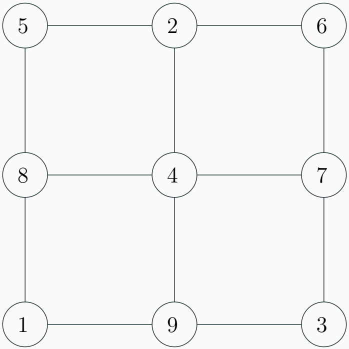

Figure 2: $F = \{1, 2, 3, 4, 5, 6\}$ and $S = \{1, 3, 4, 5, 6\}$

Observation 1

The induced graph on F obtained at the end of Step 3 in the FR14-Algorithm is a forest.

Proof: Consider any cycle C in G.

Let v be the last vertex in C in the ordering in Step 1.

Note that $v \notin F$ as both neighbors of v have been seen before v.

Thus, the induced graph of F is acyclic.

$\square$

Observations on FR14 Algorithm (contd.)

Observation 2

MWIS of the induced graph on F obtained in Step 3 in the FR14-Algorithm can be computed in linear time.

Proof: Think of dynamic programming on a rooted tree.

Consider a vertex v and let $I(v)$ represents the weight of the MWIS of the subtree rooted at v.

MWIS for the subtree rooted at v is one of the following two types:

Case 1: $v \in$ MWIS: $I(v) = wt(v) + \sum_{x \in \{\text{grandchild of } v\}} I(x)$

Case 2: $v \notin$ MWIS: $I(v) = \sum_{x \in \{\text{child of } v\}} I(x)$

□

Claim

The weight of the independent S returned by the FR14-Algorithm satisfies $E\left[\sum_{v\in S} w(v)\right] \geq 2 \cdot \sum_{v\in I^*} \frac{w(v)}{deg(v)+1}$, where I^* is a maximum weight independent set of G.

Proof: Let I be an independent set of G.

Observe that $I \cap F$ is an independent set of induced graph of F.

Since S is a MWIS of the induced graph of F (see Step 4), we have

$$E\left[\sum_{v\in S} w(v)\right] \geq E\left[\sum_{v\in I\cap F} w(v)\right]$$

Consider a vertex $v \in I$.

When does v makes contribution to the sum $E\left[\sum_{v\in I\cap F} w(v)\right]$?

Analysis of FR14 Algorithm (contd.)

When does v makes contribution to the sum $E\left[\sum_{v\in I\cap F} w(v)\right]$?

Only if, it is included in F.

$Pr(v \in F) = \frac{2}{deg(v)+1}$ (it has to be either the 1st or the 2nd vertex among its neighbors in the permutation ordering to be included in F)

We have $E\left[\sum_{v\in I\cap F} w(v)\right] = E\left[\sum_{v\in I} w(v)X_v\right]$, where X_v is indicator r.v. stating whether $v \in F$ or $v \notin F$.

Thus, $E\left[\sum_{v\in S} w(v)\right] \geq E\left[\sum_{v\in I} w(v)X_v\right] = \sum_{v\in I} w(v)E\left[X_v\right] = \sum_{v\in I} w(v)\frac{2}{deg(v)+1}$

Observe that we can replace the independent set I by the MWIS I^* of G, and we have $E\left[\sum_{v\in S} w(v)\right] \geq 2 \cdot \sum_{v\in I^*} \frac{w(v)}{deg(v)+1}$

□

References

References

1. U. Feige and D. Reichman, Recoverable values for independent sets. Random Structures & Algorithms, 2014.
2. Johan Hastad, Some optimal inapproximability results. Journal of the ACM, 48(4):798-859, 2001.
3. Tim Roughgarden, Beyond Worst Case Analysis Lecture Notes, 2014.