Densest subgraphs, iterative peeling, and supermodularity

November 20, 2024

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COMP 5112 Project Presentation

Outline

1. Densest subgraph problem (**DSP**): motivation and definitions
2. Peeling, iterative peeling for graphs (**Greedy++**); associated guarantees
3. Densest supermodular set problem (**DSSP**)
   i. Iterative peeling for DSSP (**SuperGreedy++**)
   ii. **Convergence** of iterative peeling for DSSP
4. References

This presentation is primarily based on the work of Boob et. al (2019)

Flowless: Extracting Densest Subgraphs Without Flow Computations

and the work of Chekuri, Quanrud, Torres (2022)

Densest Subgraph: Supermodularity, Iterative Peeling, and Flow

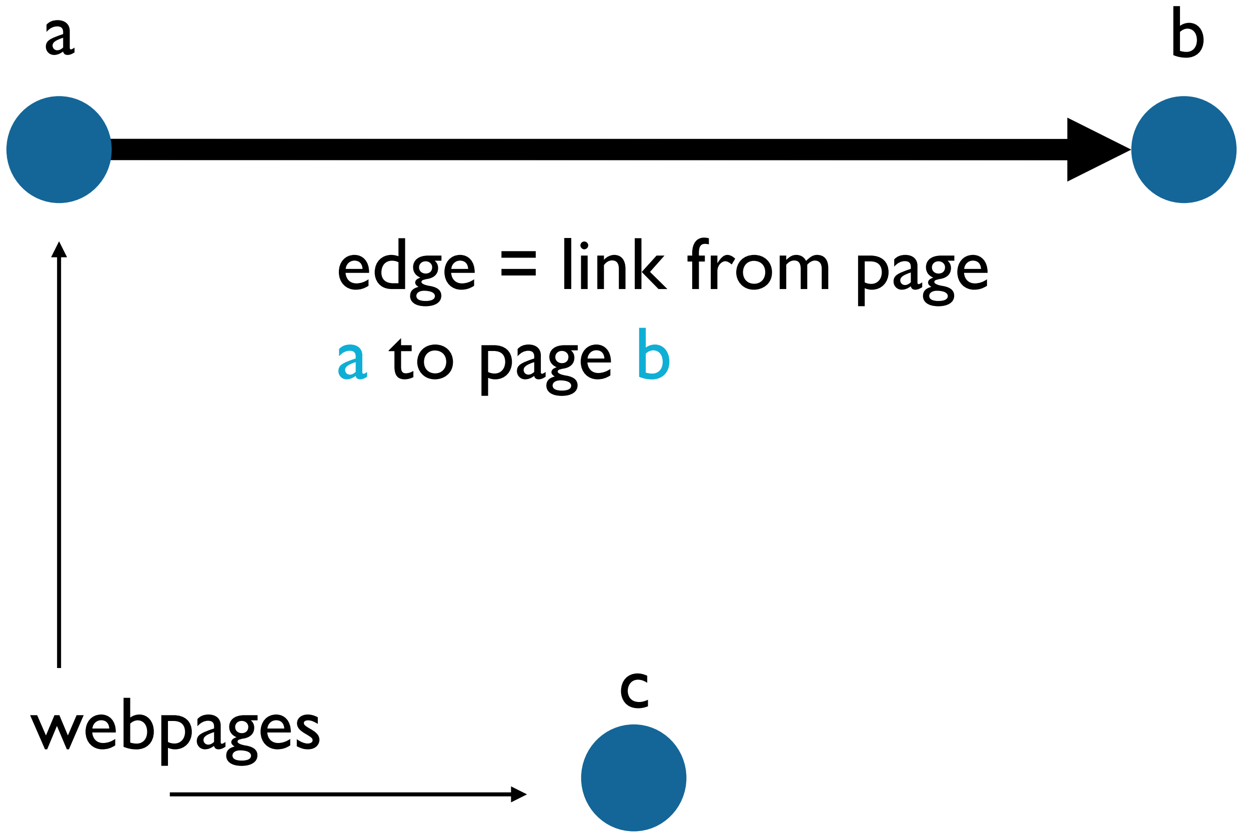
Part 1:

The Densest Subgraph Problem (DSP)

(motivation and definitions)

Many real-world problems can be formulated as finding “clusters” in graphs or optimizing “density measures” on graphs.

community detection



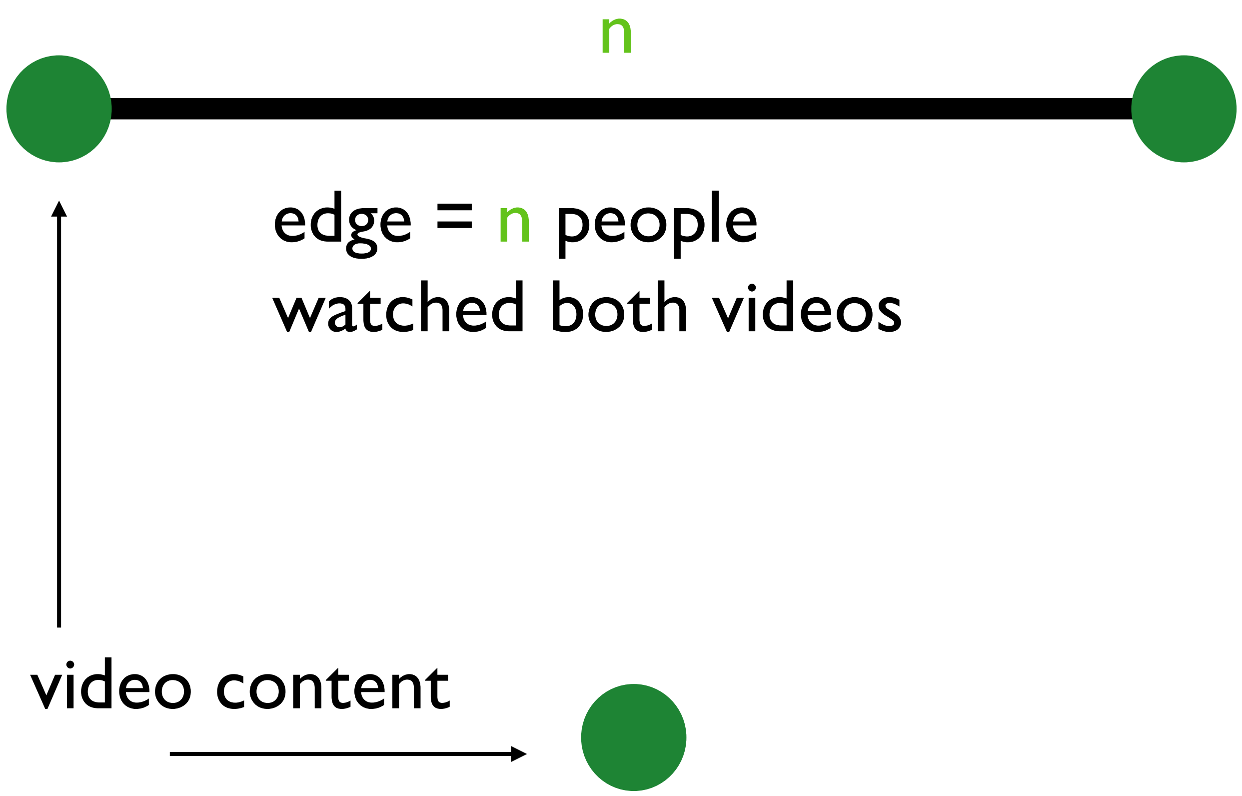
(Kleinberg, 1999)

community detection

- how do we detect groups of web pages that are related to each other?
- how do we find the "authoritative" web pages?

(Kleinberg, 1999)

topic clustering



(Tsourakakis, Chen, SDM 2021)

topic clustering

- how do we decide which content is similar?
- which groups of related content are the most popular?

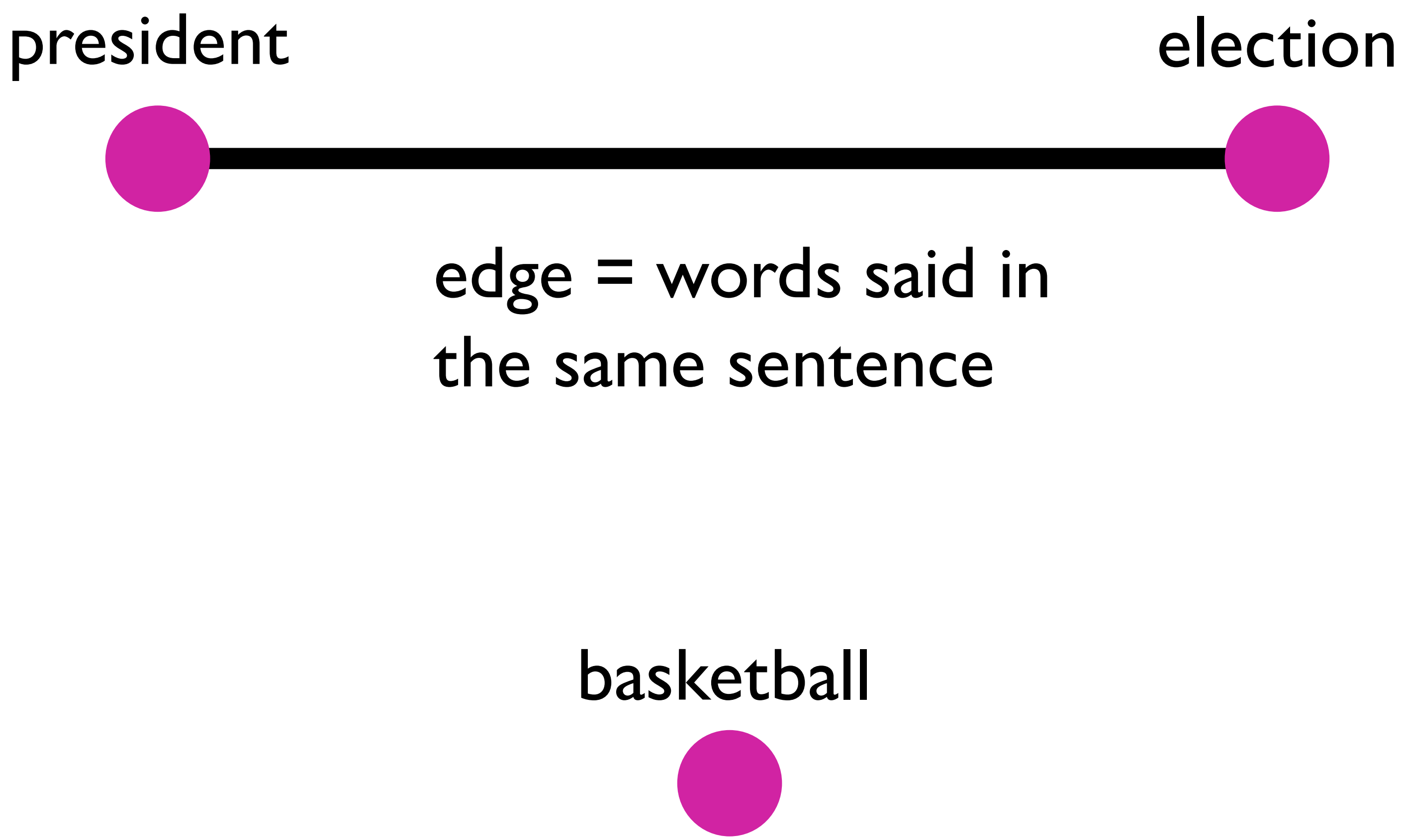
(Tsourakakis, Chen, SDM 2021)

topic clustering

- how do we decide which content is similar?
- which groups of related content are the most popular?
- “large near-clique extraction” based on “thematic coherence”

(Tsourakakis, Chen, SDM 2021)

“text networks”



(Chen, Saad, 2012)
(Corman et. al, 2002)

“text networks”

- which groups of words appear most frequently together?
- what is the topic of the text?

(Chen, Saad, 2012)
(Corman et. al, 2002)

“text networks”

- which groups of words appear most frequently together?
- what is the topic of the text?
- used in linguistics
- “centering resonance analysis”

(Chen, Saad, 2012)
(Corman et. al, 2002)

This has motivated the study of associated techniques:

- correlation mining (finance, neuroscience, genetics, etc.)

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- correlation mining (finance, neuroscience, genetics, etc.)
- graph clustering, graph compression
- "dense subgraph discovery" (there are multiple variants)

In the broadest sense, the DSP asks us to find the subgraph that maximizes some measure of density.

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- **input:** a graph G, a density function f
- **output:** the subgraph of G with the highest density under f

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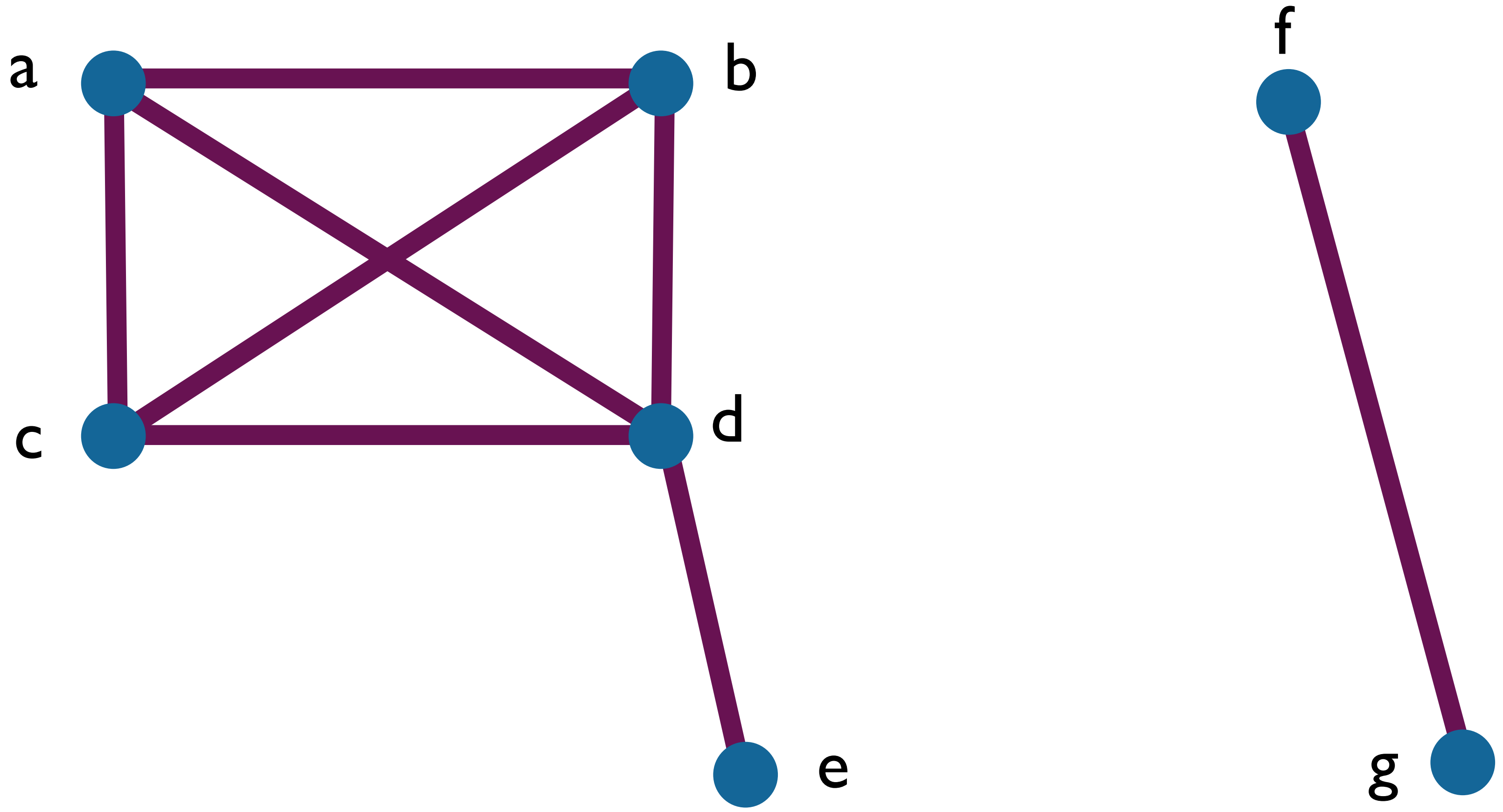
- input: a graph G, a density function f
- output: the subgraph of G with the highest density under f

Formally, we often find the subset of the vertices that induces the densest subgraph.

graph theory (definitions)

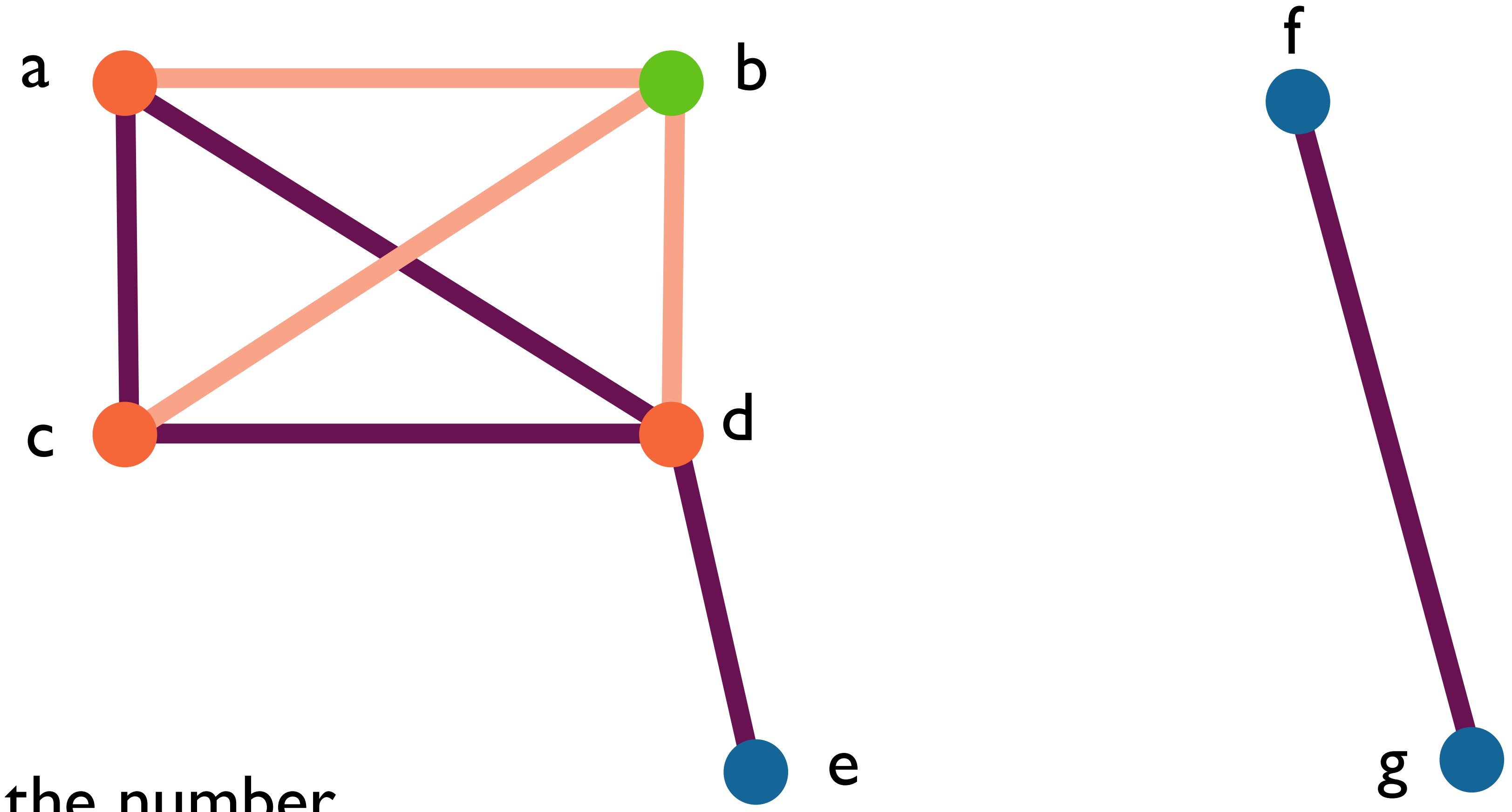


G = (V, E)

order = |V| = n = 7

size = |E| = m = 8

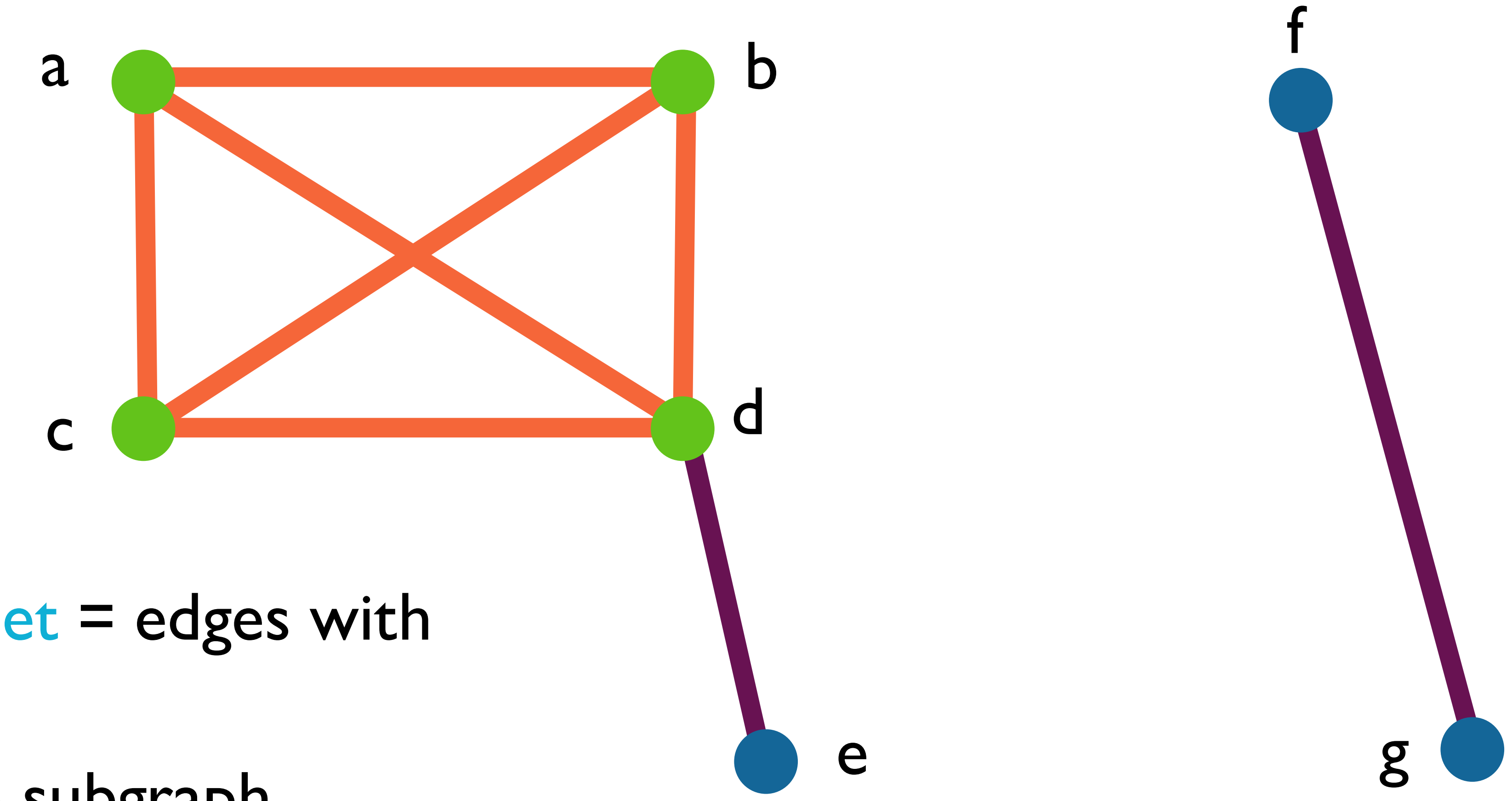
graph theory (definitions)



deg(b) = |nbr(b)| = 3

alternatively, deg(b) is the number of edges connected to b

graph theory (definitions)



S = {a, b, c, d}

E(S) = induced edge set = edges with both endpoints in S

G[S] = (S, E(S)) is the subgraph induced by S

the densest subgraph problem

The density of a graph is the number of edges it has divided by the number of vertices.

We want to find the subgraph that induces the highest density (largest average degree).

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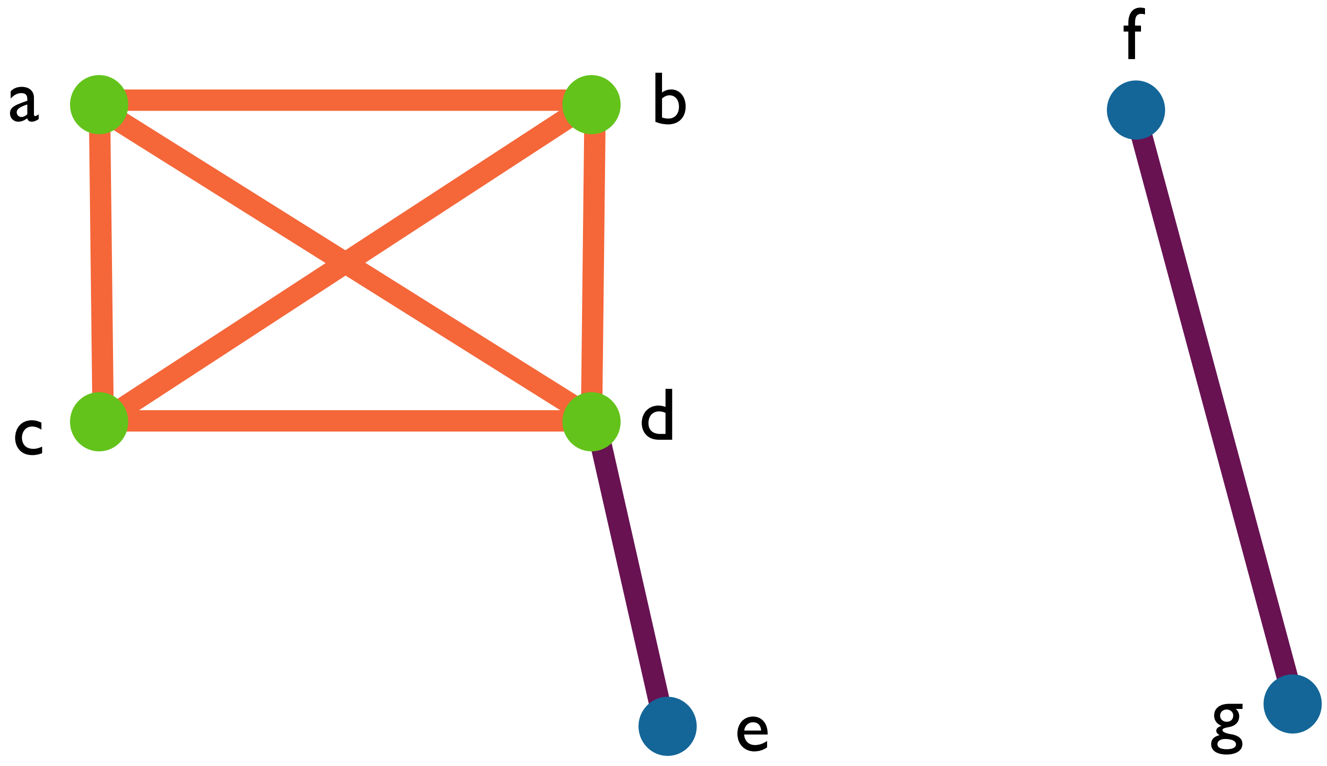


Here, the densest subgraph is induced by the set S = {a, b, c, d}.

The DSP, formally.

Given a graph $G = (V, E)$, let $S \subseteq V$. Then $G[S] = (S, E(S))$ and

$$density(S) = \frac{|E(S)|}{|S|}$$

The DSP, formally.

Given a graph $G = (V, E)$, let $S \subseteq V$. Then $G[S] = (S, E(S))$ and

$$density(S) = \frac{|E(S)|}{|S|}$$

We want to find the subgraph with the optimal density, λ^*:

$$\lambda^* = \max_{S \subseteq V} \frac{|E(S)|}{|S|}$$

Part 2:

Iterative peeling for graphs (Greedy++)

(history, algorithms, and associated results)

peeling: an algorithm

- Given a graph, repeatedly remove the vertex with the current lowest degree, as well as all edges attached to it.

(Asahiro et. al, 1996; Charikar, 2000)

peeling: an algorithm

- Given a graph, repeatedly remove the vertex with the current lowest degree, as well as all edges attached to it.
- From this, we get an ordering $v_1, v_2, \ldots, v_n$ of vertices, where v_i is the i_{th} vertex in the removal order.

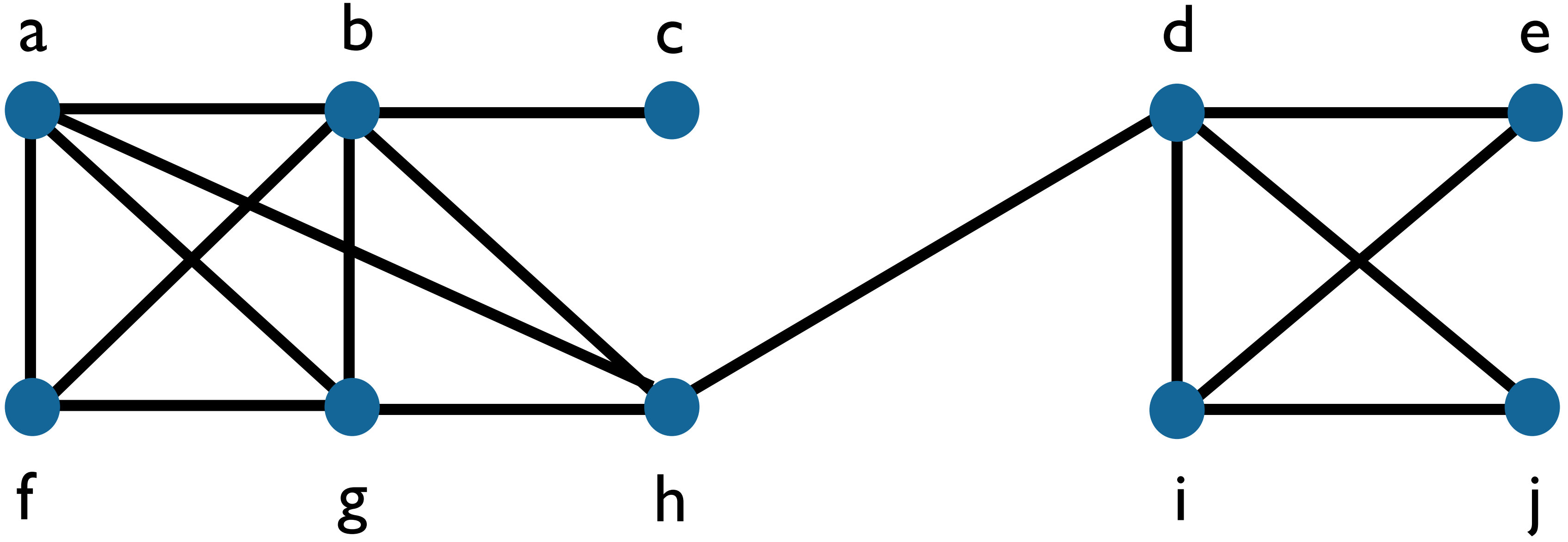
(Asahiro et. al, 1996; Charikar, 2000)

peeling: an algorithm

- Given a graph, repeatedly remove the vertex with the current lowest degree, as well as all edges attached to it.
- From this, we get an ordering $v_1, v_2, \ldots, v_n$ of vertices, where v_i is the i_{th} vertex in the removal order.
- We choose the suffix $S_i = \{v_i, v_{i+1}, \ldots, v_n\}$ that induces the subgraph with the highest density λ.

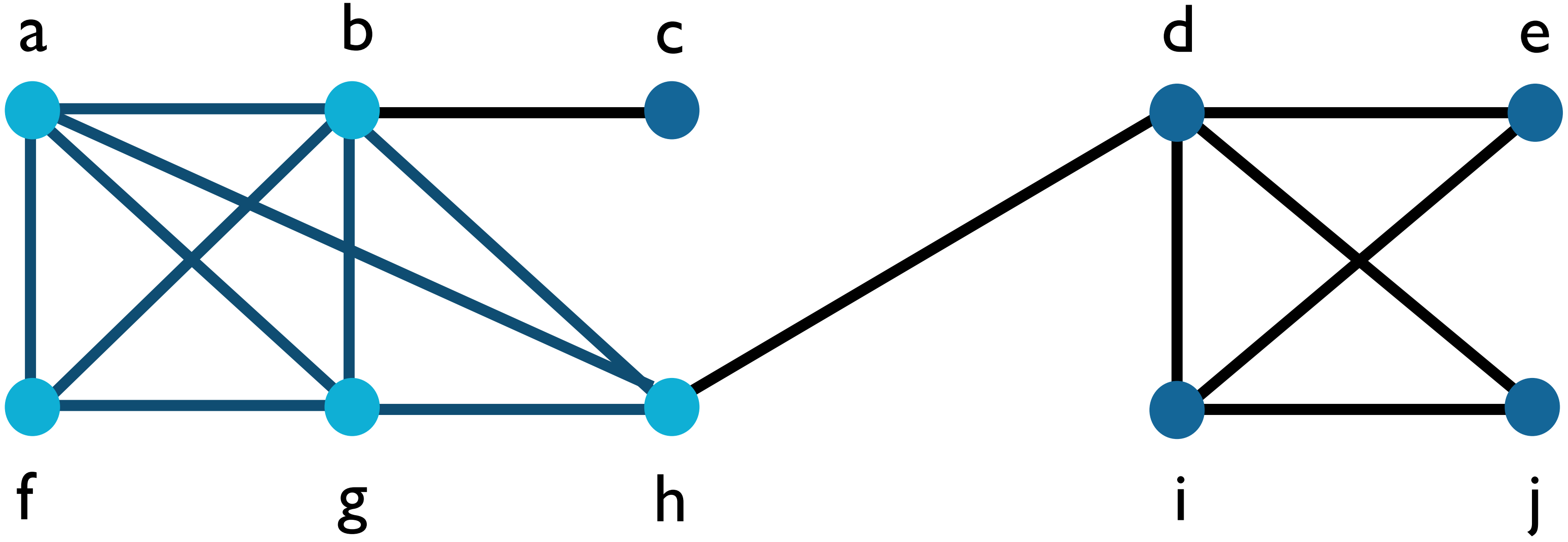
(Asahiro et. al, 1996; Charikar, 2000)

peeling: an algorithm

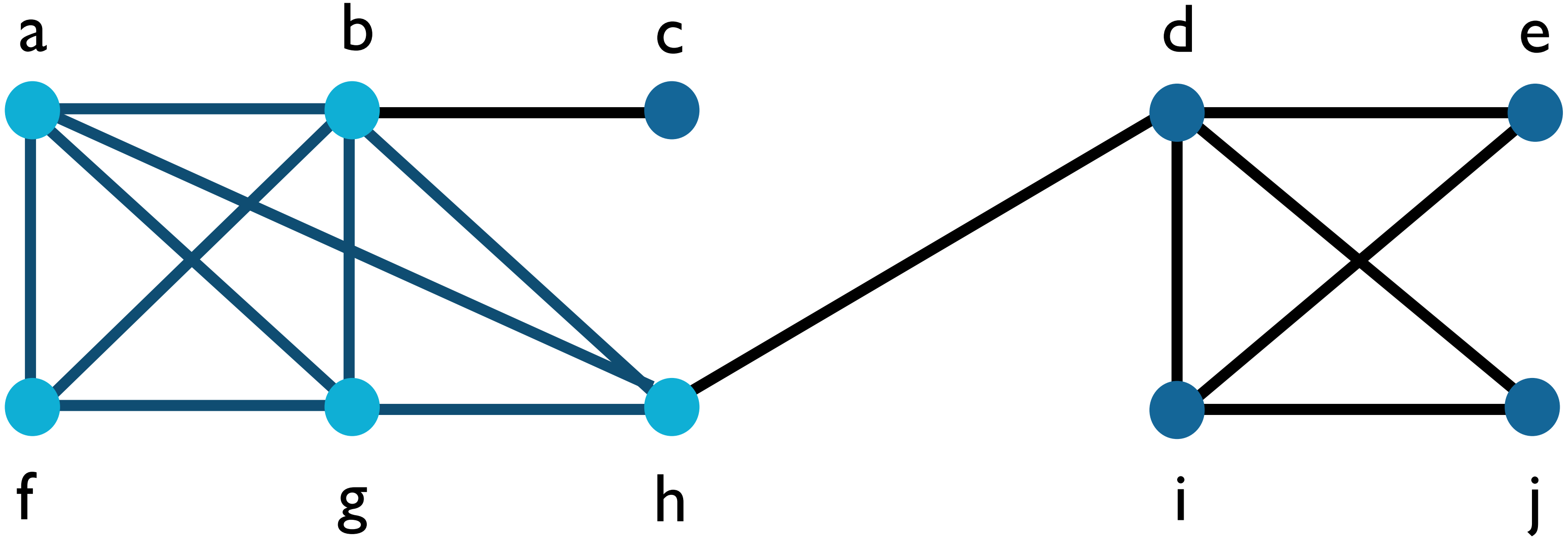


peeling: an algorithm

optimal density:
9/5 = 1.8



peeling: an algorithm



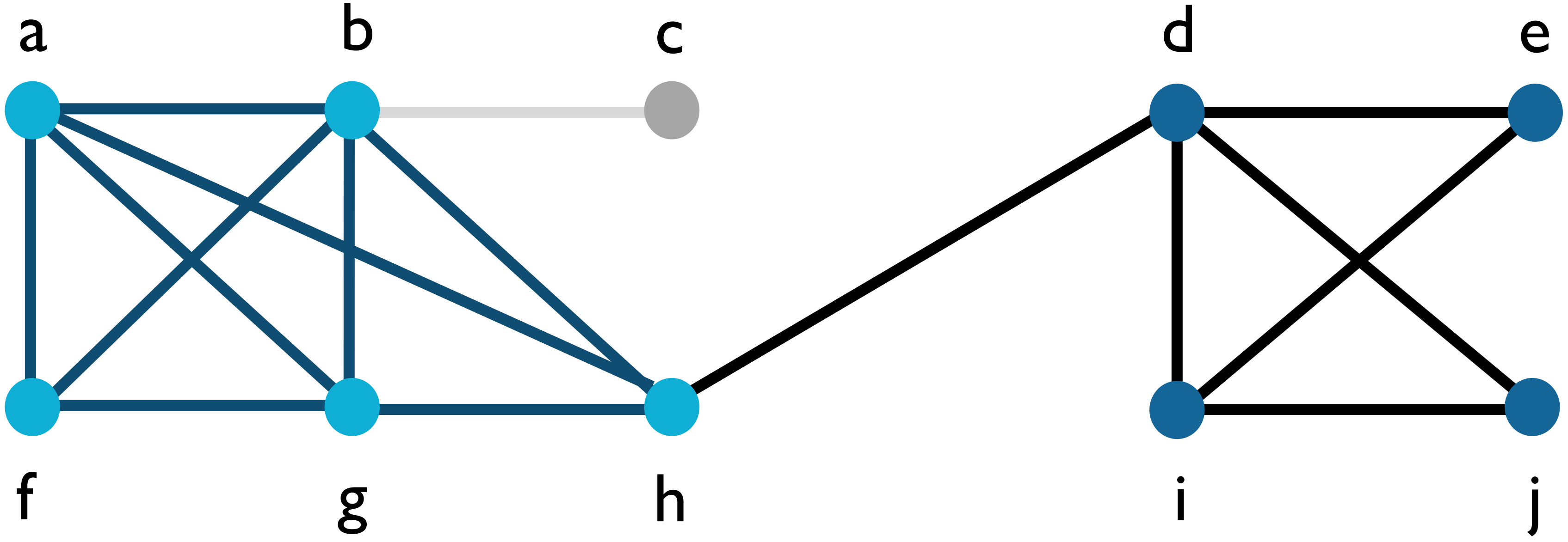
optimal density:
9/5 = 1.8

current density:
16/10 = 1.6

highest density:
16/10 = 1.6

	a	b	c	d	e	f	g	h	i	j
degree	4	5	1	4	2	3	4	4	3	2
curr. degree	4	5	1	4	2	3	4	3	3	2
final degree										

peeling: an algorithm



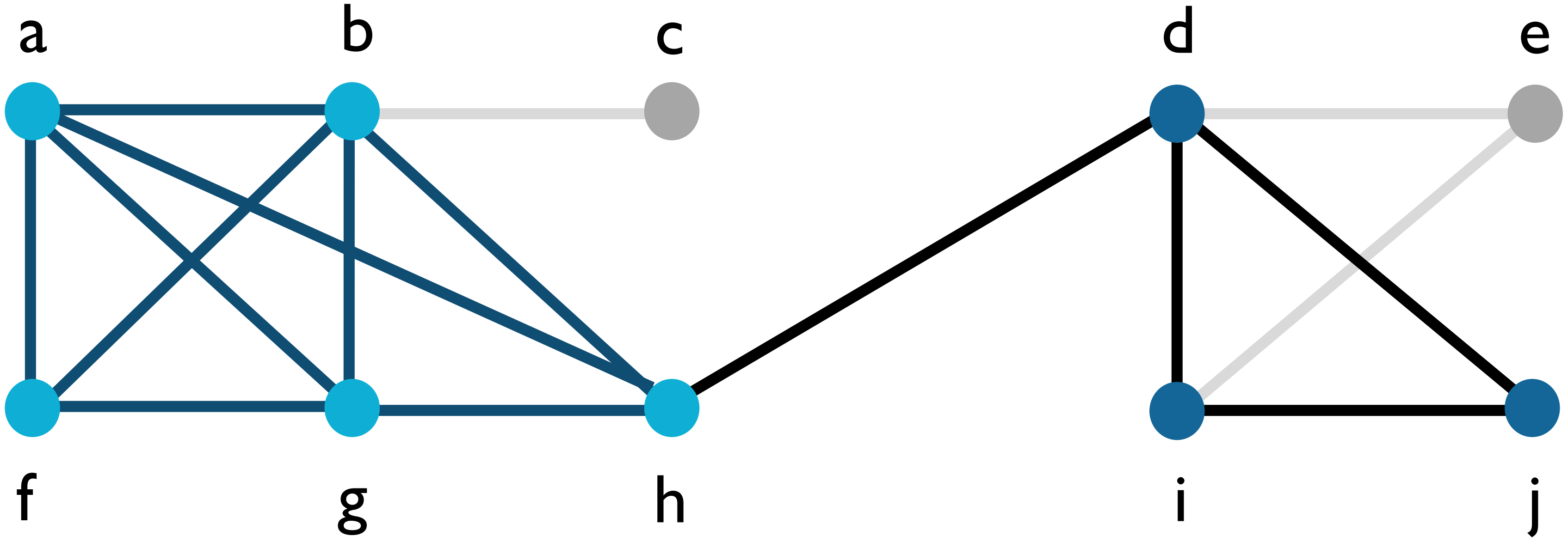
optimal density:
9/5 = 1.8

current density:
15/9 = 1.6666667

highest density:
15/9 = 1.6666667

a	b	c	d	e	f	g	h	i	j	
4	5	1	4	2	3	4	4	3	2	degree
4	4	X	4	2	3	4	3	3	2	curr. degree
		1								final degree

peeling: an algorithm



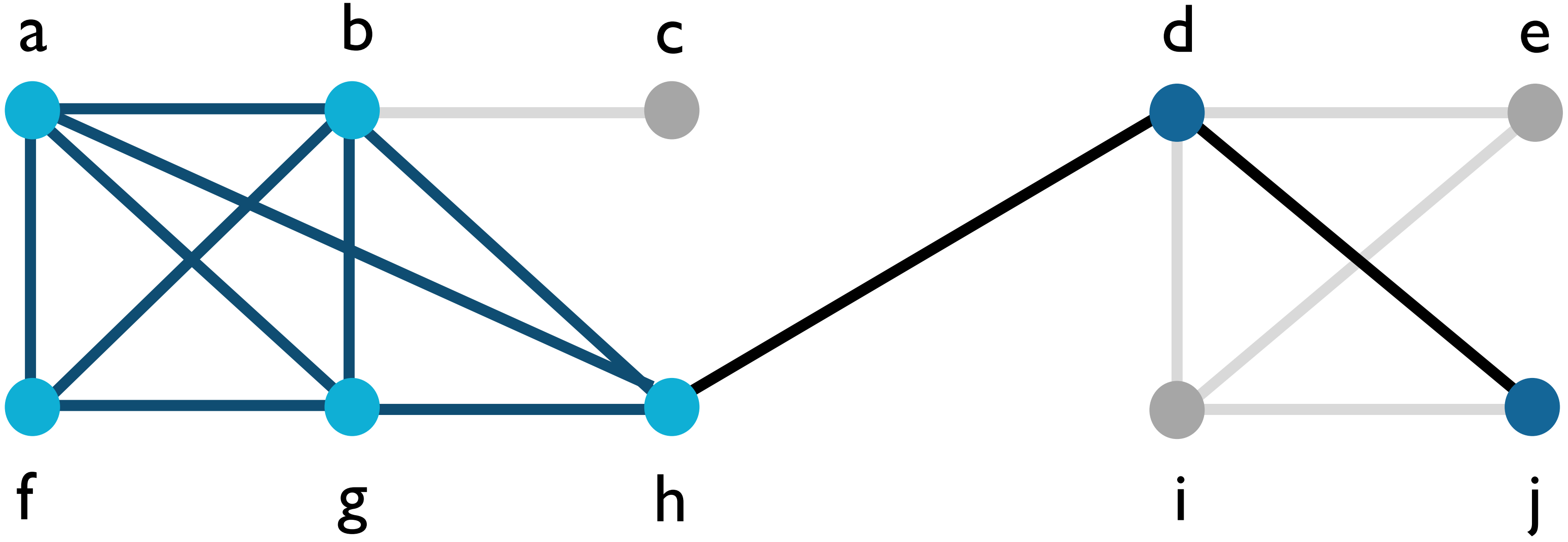
optimal density:
9/5 = 1.8

current density:
13/8 = 1.625

highest density:
15/9 = 1.6666667

a	b	c	d	e	f	g	h	i	j	
4	5	1	4	2	3	4	4	3	2	degree
4	4	X	3	X	3	4	3	2	2	curr. degree
		1		2						final degree

peeling: an algorithm



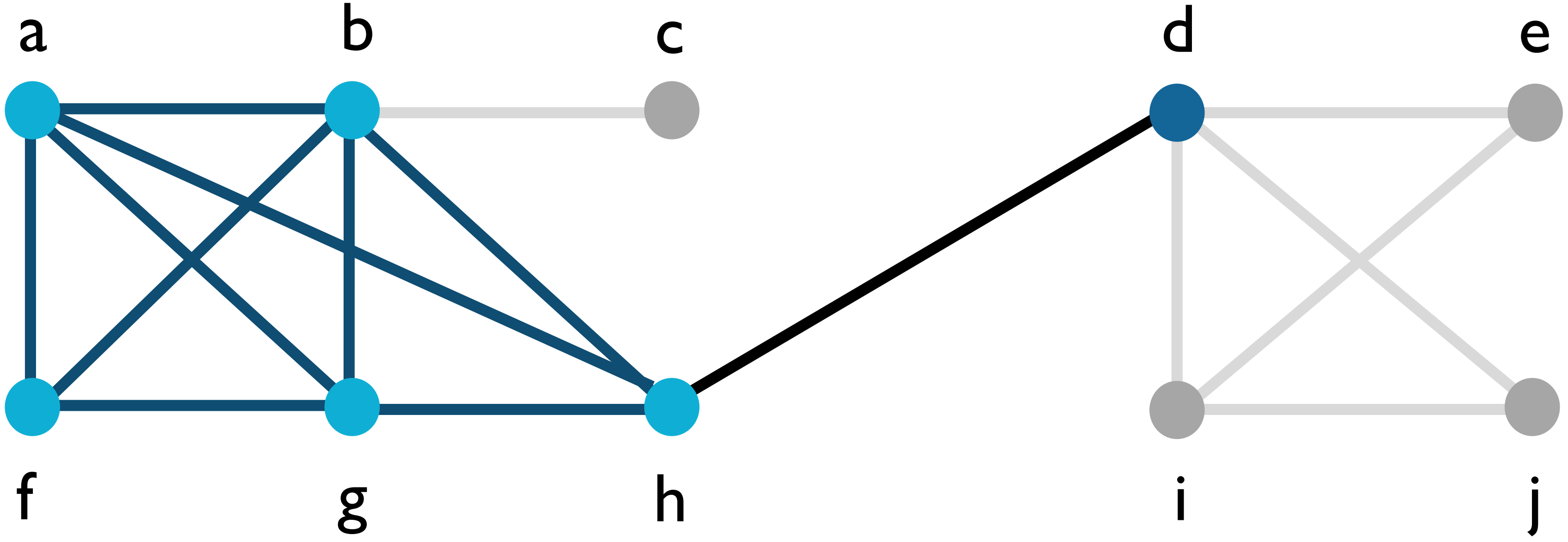
optimal density:
9/5 = 1.8

current density:
11/7 = 1.625

highest density:
15/9 = 1.6666667

a	b	c	d	e	f	g	h	i	j	
4	5	1	4	2	3	4	4	3	2	degree
4	4	X	2	X	3	4	3	X	1	curr. degree
		1		2				2		final degree

peeling: an algorithm



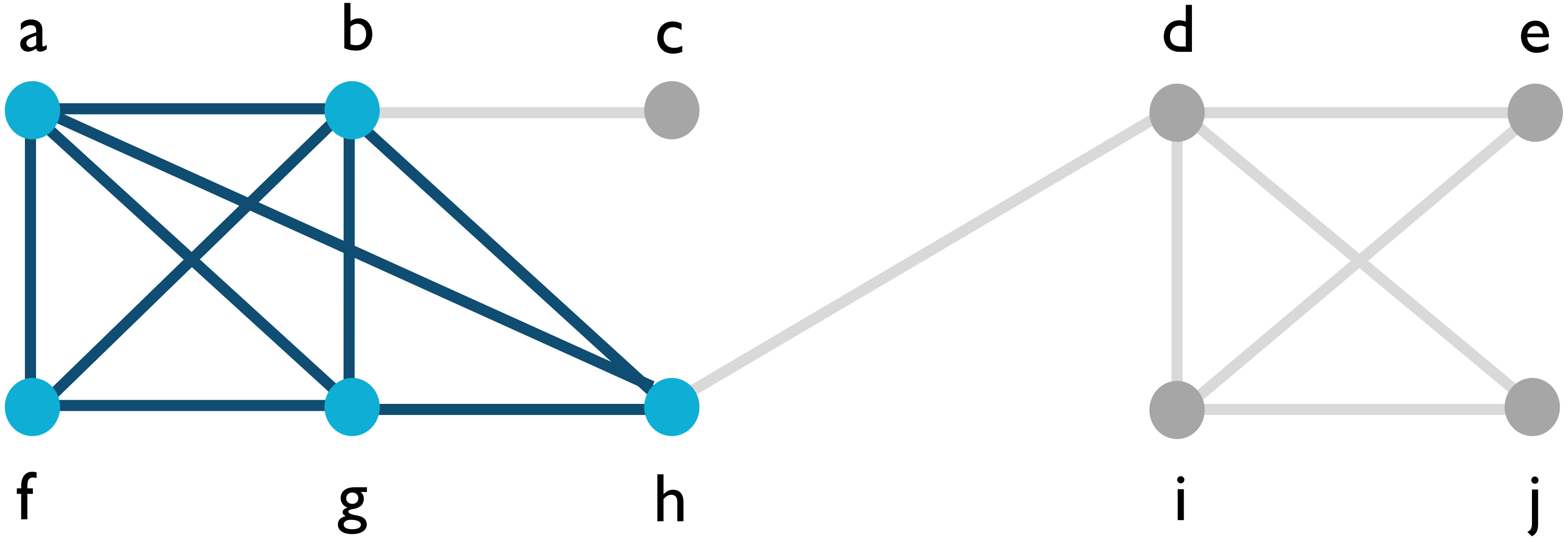
optimal density:
9/5 = 1.8

current density:
10/6 = 1.6666667

highest density:
15/9 = 1.6666667

a	b	c	d	e	f	g	h	i	j	
4	5	1	4	2	3	4	4	3	2	degree
4	4	X	1	X	3	4	3	X	X	curr. degree
		1		2				2	1	final degree

peeling: an algorithm



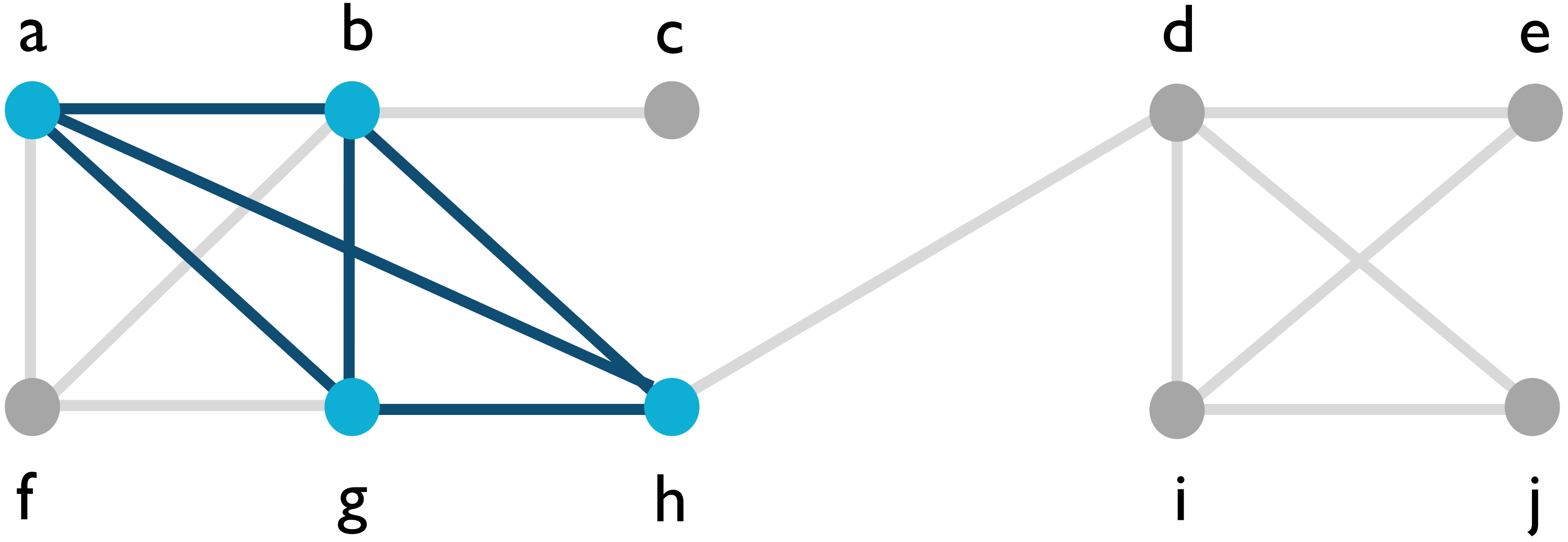
optimal density:
9/5 = 1.8

current density:
9/5 = 1.8

highest density:
9/5 = 1.8

a	b	c	d	e	f	g	h	i	j	
4	5	1	4	2	3	4	4	3	2	degree
4	4	X	X	X	3	4	3	X	X	curr. degree
		1	1	2				2	1	final degree

peeling: an algorithm



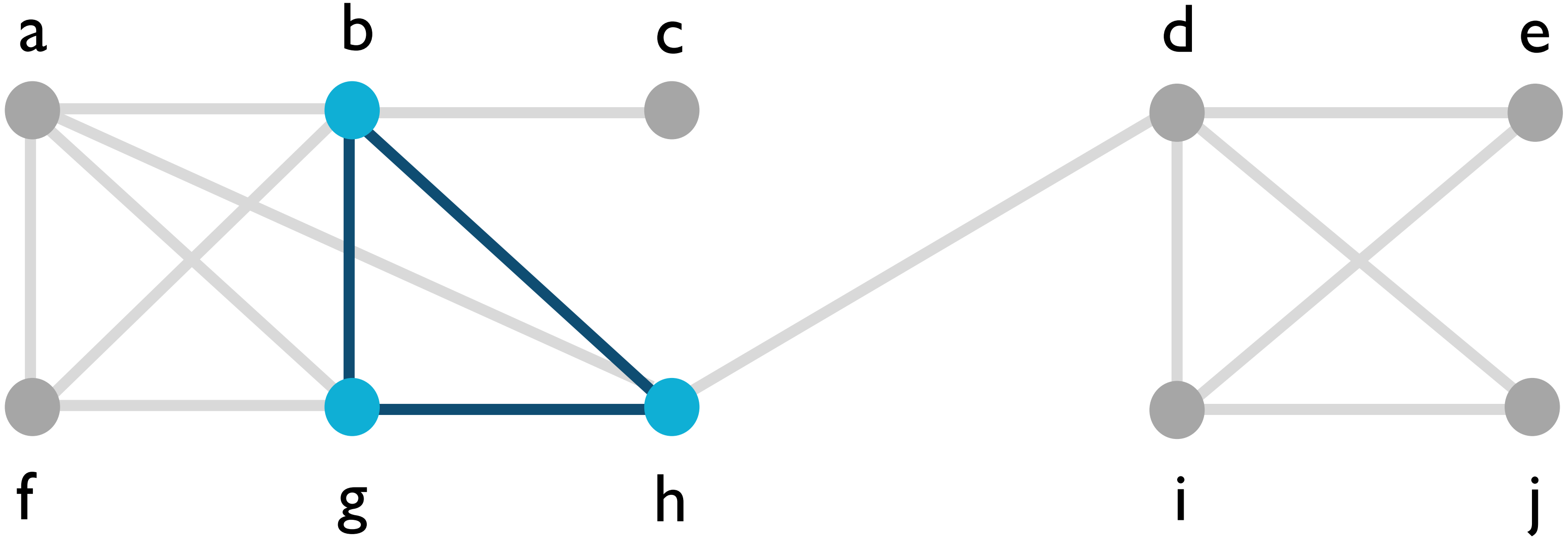
optimal density:
9/5 = 1.8

current density:
6/4 = 1.5

highest density:
9/5 = 1.8

a	b	c	d	e	f	g	h	i	j	
4	5	1	4	2	3	4	4	3	2	degree
3	3	X	X	X	X	3	3	X	X	curr. degree
		1	1	2	3			2	1	final degree

peeling: an algorithm



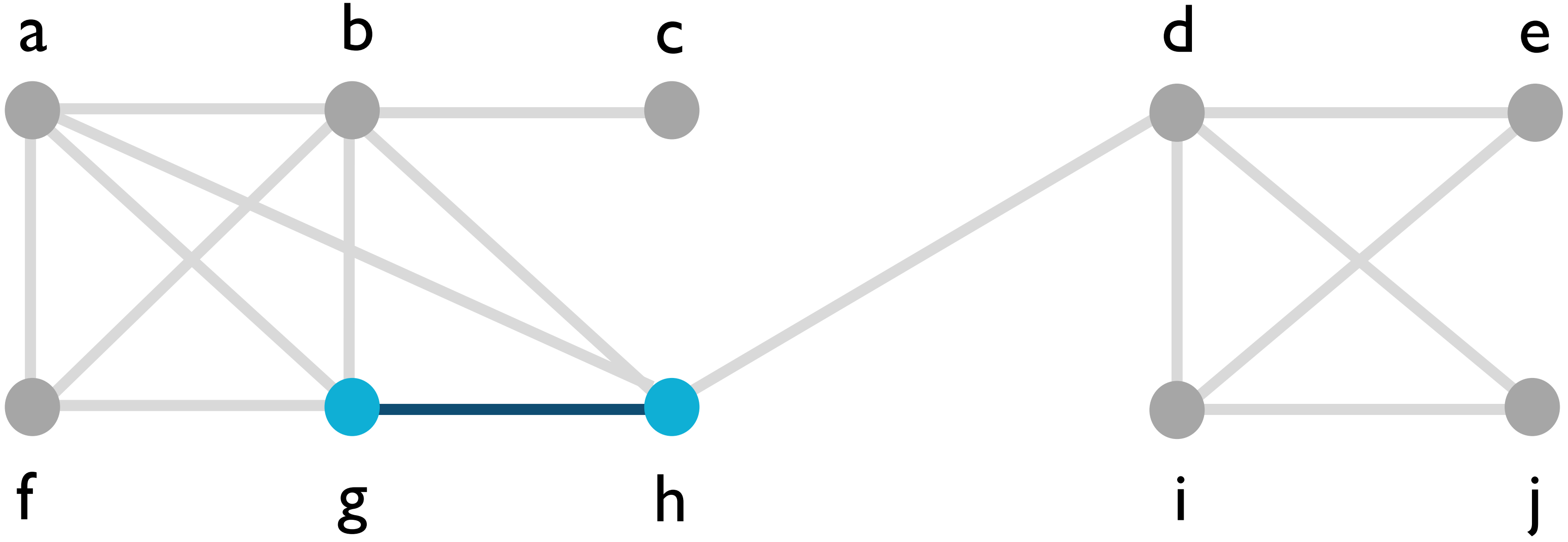
optimal density:
9/5 = 1.8

current density:
3/3 = 1

highest density:
9/5 = 1.8

a	b	c	d	e	f	g	h	i	j	
4	5	1	4	2	3	4	4	3	2	degree
X	2	X	X	X	X	2	2	X	X	curr. degree
3		1	1	2	3			2	1	final degree

peeling: an algorithm



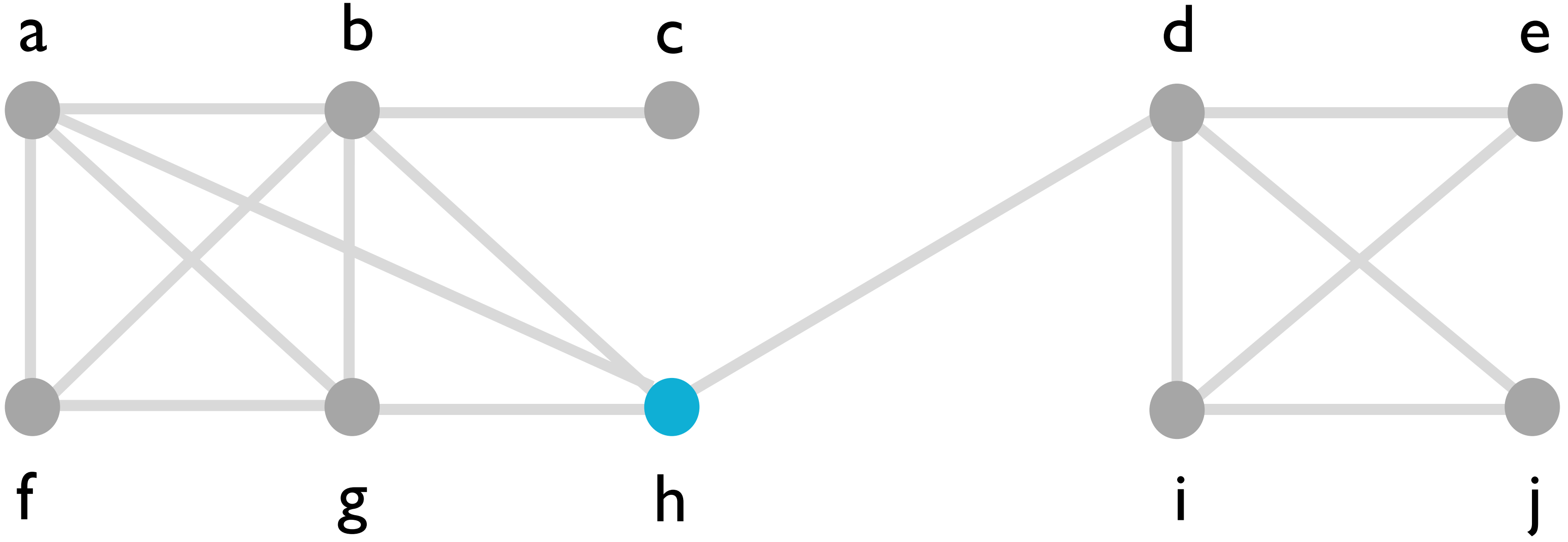
optimal density:
9/5 = 1.8

current density:
1/2 = 0.5

highest density:
9/5 = 1.8

a	b	c	d	e	f	g	h	i	j	
4	5	1	4	2	3	4	4	3	2	degree
X	X	X	X	X	X	1	1	X	X	curr. degree
3	2	1	1	2	3			2	1	final degree

peeling: an algorithm



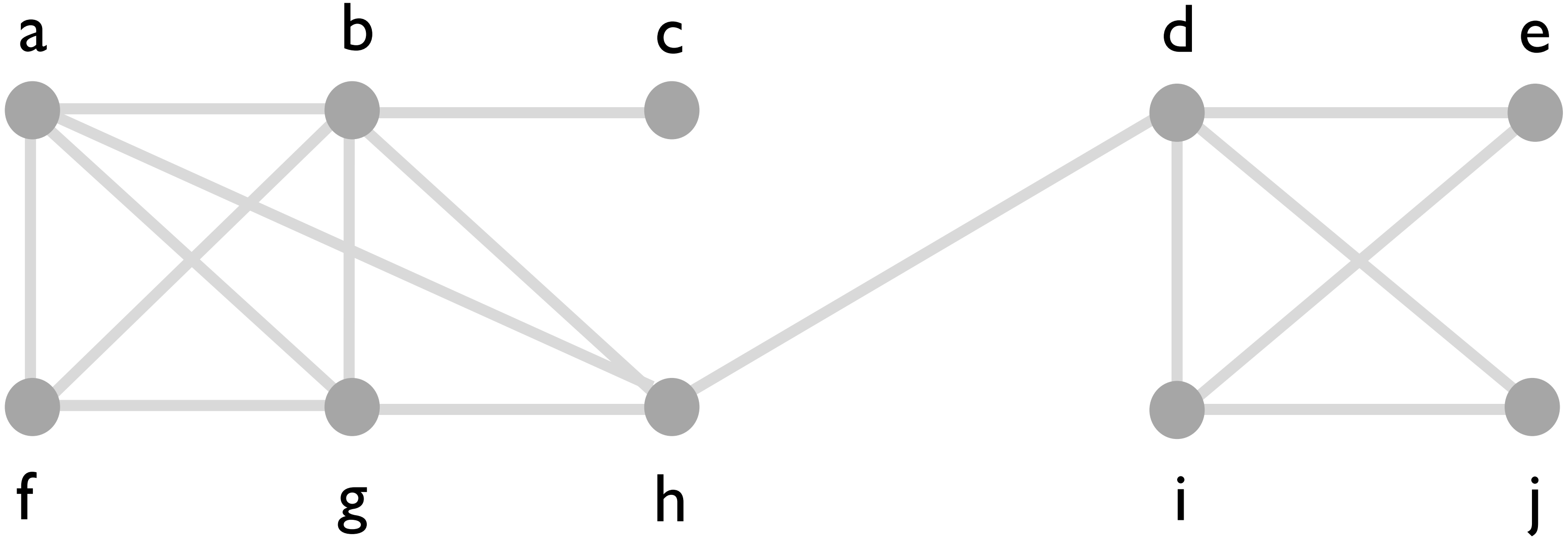
optimal density:
9/5 = 1.8

current density:
0/1 = 0

highest density:
9/5 = 1.8

a	b	c	d	e	f	g	h	i	j	
4	5	1	4	2	3	4	4	3	2	degree
X	X	X	X	X	X	X	0	X	X	curr. degree
3	2	1	1	2	3	1		2	1	final degree

peeling: an algorithm



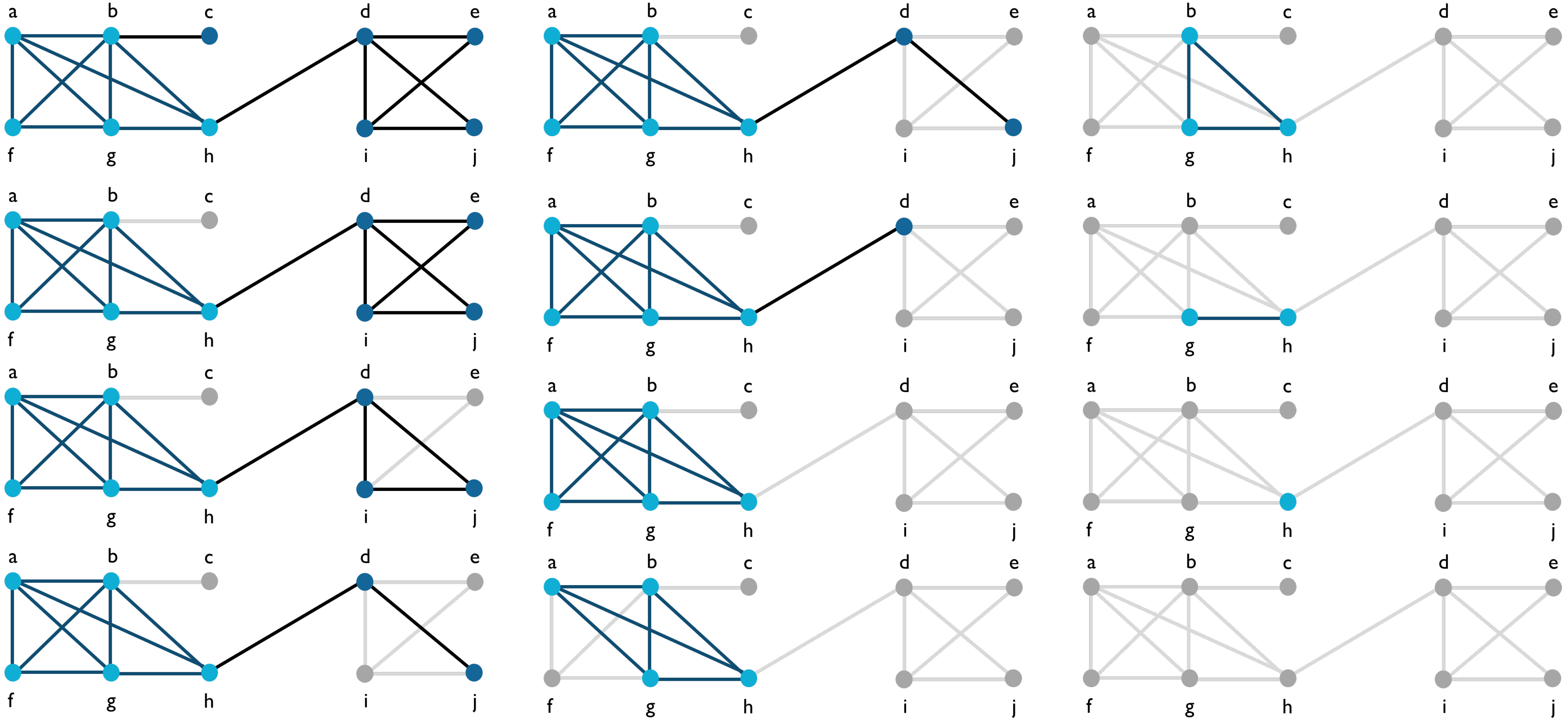
optimal density:
9/5 = 1.8

current density:
0/0 = undefined

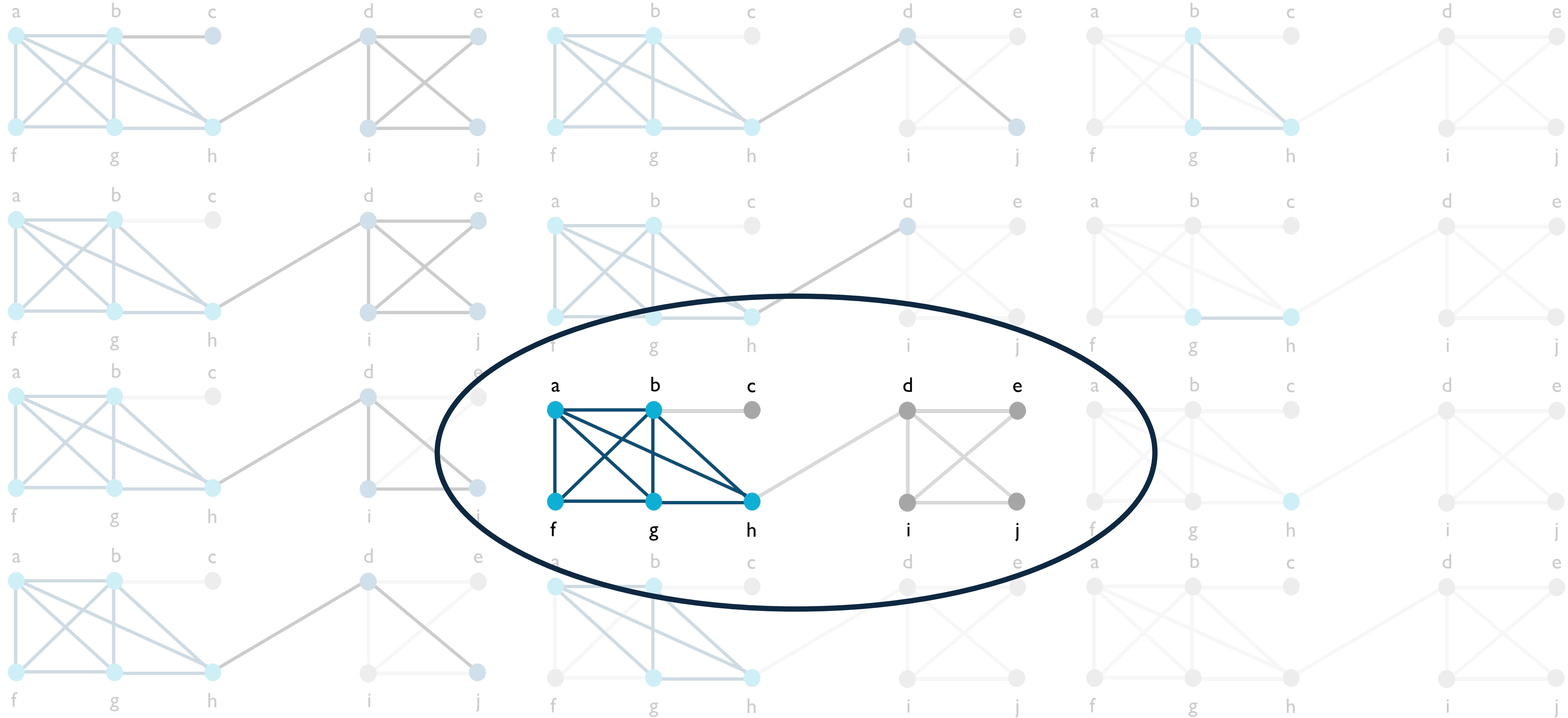
highest density:
9/5 = 1.8

a	b	c	d	e	f	g	h	i	j	
4	5	1	4	2	3	4	4	3	2	degree
X	X	X	X	X	X	X	1	X	X	curr. degree
3	2	1	1	2	3	1	0	2	1	final degree

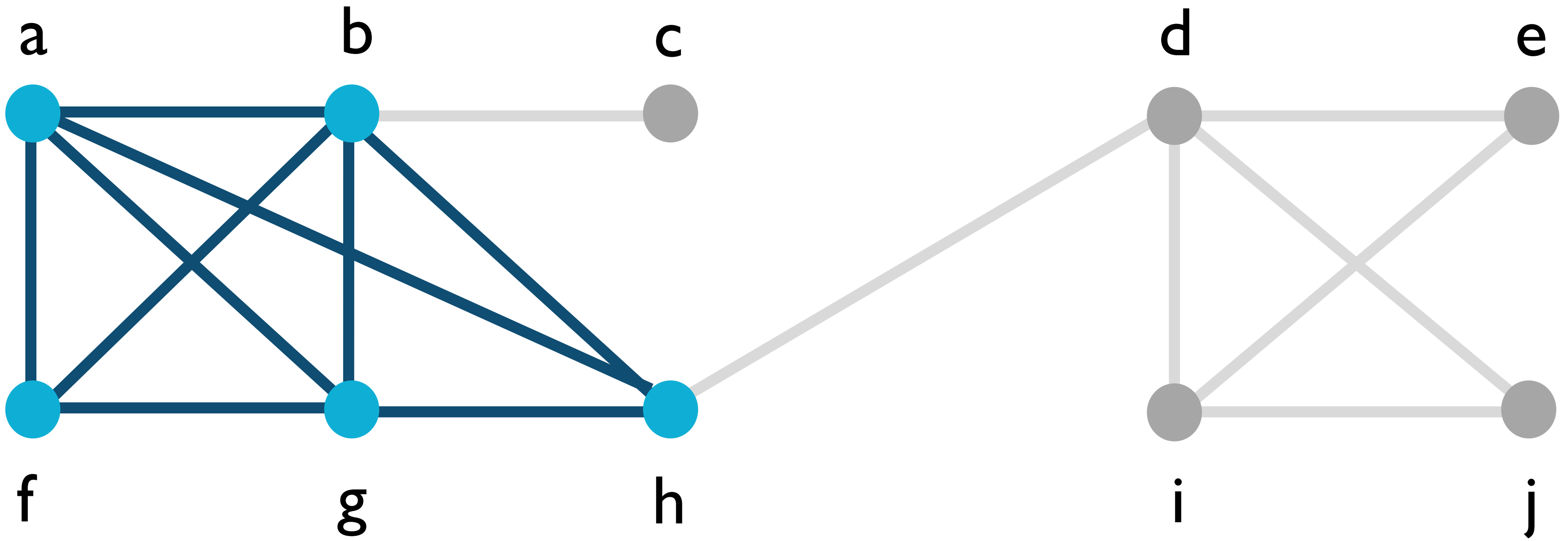
peeling: an algorithm



peeling: an algorithm



peeling: an algorithm



In this case, the algorithm did produce the expected densest subgraph. However, this is not always the case.

peeling: summary

- very fast! (runs in linear time)
- ½-approximation for DSP (Charikar, 2000)
- *usually* about 80% good on real world graphs
- used in practice (real applications)
- how can we do better?

iterative peeling (Greedy++)

- Decide on a number of iterations T;

(Boob et. al, 2019)

iterative peeling (Greedy++)

- Decide on a number of iterations T;
- Keep an array of loads for each vertex, all initialized to 0. Let the load of v during iteration i = $\mathrm{load}_i(v)$;

(Boob et. al, 2019)

iterative peeling (Greedy++)

- Decide on a number of iterations T;
- Keep an array of loads for each vertex, all initialized to 0. Let the load of v during iteration i = $\mathrm{load}_i(v)$;
- In each iteration i, repeatedly find the vertex that minimizes $\mathrm{current\_deg}(v) + \mathrm{load}_{i-1}(v)$, and remove it;

(Boob et. al, 2019)

iterative peeling (Greedy++)

- Decide on a number of iterations T;
- Keep an array of loads for each vertex, all initialized to 0. Let the load of v during iteration i = $\mathrm{load}_i(v)$;
- In each iteration i, repeatedly find the vertex that minimizes $\mathrm{current\_deg}(v) + \mathrm{load}_{i-1}(v)$, and remove it;
- Let $\mathrm{load}_i(v)$ = $\mathrm{load}_{i-1}(v)$ + the degree of v when it was removed;

(Boob et. al, 2019)

iterative peeling (Greedy++)

- Let $v_{1,j}\ , v_{2,j}\ , \ldots , v_{n,j}$ be the ordering from vertex removal during iteration j;

(Boob et. al, 2019)

iterative peeling (Greedy++)

- Let $v_{1,j}\ , v_{2,j}\ , \ldots , v_{n,j}$ be the ordering from vertex removal during iteration j;
- After T iterations, we choose the suffix over all orderings $S_{i,j} = \{v_{i,j},\ v_{i+1,j}, \ldots , v_{n,j}\}$ that induces the subgraph with the highest density λ;

(Boob et. al, 2019)

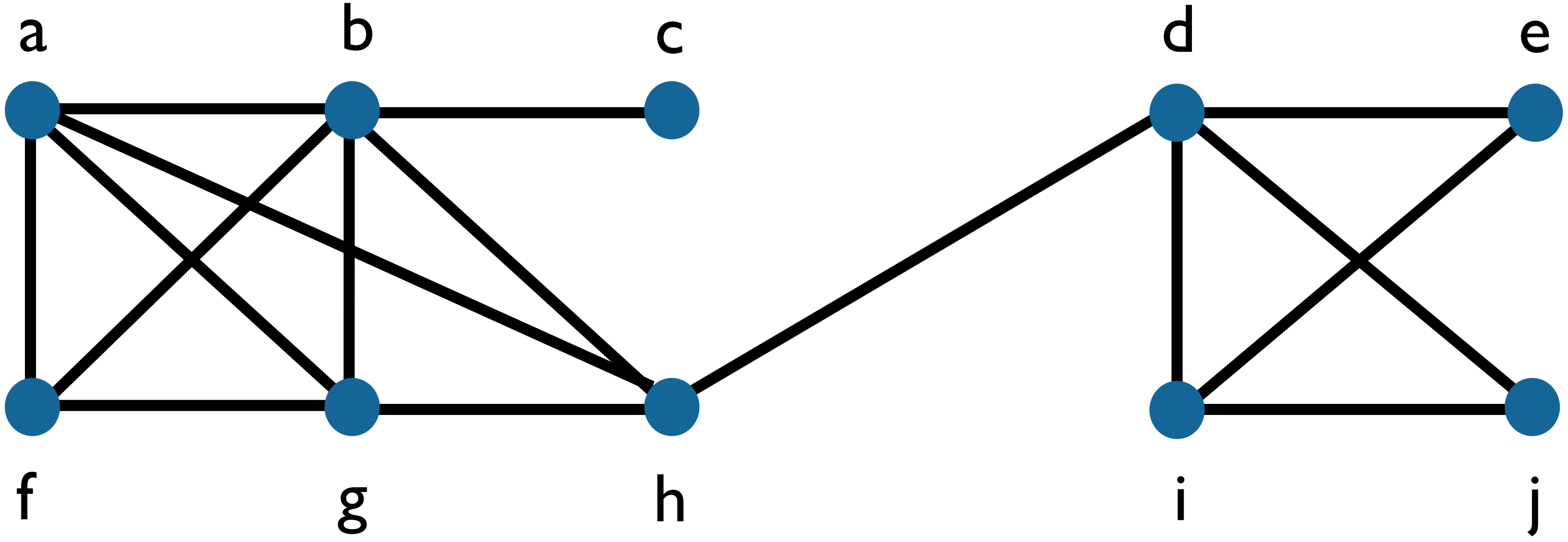
iterative peeling (Greedy++)

- Let $v_{1,j}\,, v_{2,j}\,, \ldots, v_{n,j}$ be the ordering from vertex removal during iteration j;
- After T iterations, we choose the suffix over all orderings $S_{i,j} = \{v_{i,j},\, v_{i+1,j}, \ldots, v_{n,j}\}$ that induces the subgraph with the highest density λ;
- Typically, this is a suffix $S_{i,T} = \{v_{i,T},\, v_{i+1,T}, \ldots, v_{n,T}\}$.

(Boob et. al, 2019)

iterative peeling

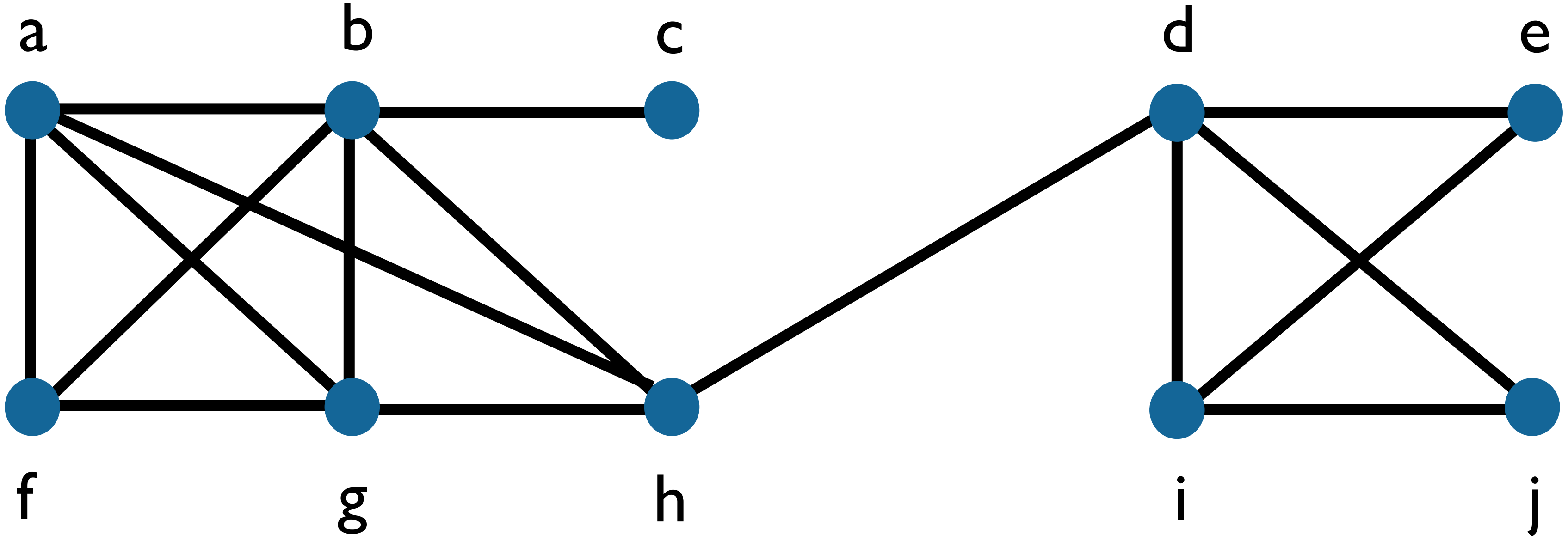
(Boob et. al, 2019)



a	b	c	d	e	f	g	h	i	j	
4	5	1	4	2	3	4	4	3	2	degree
X	X	X	X	X	X	X	1	X	X	curr. degree
3	2	1	1	2	3	1	0	2	1	final degree

iterative peeling

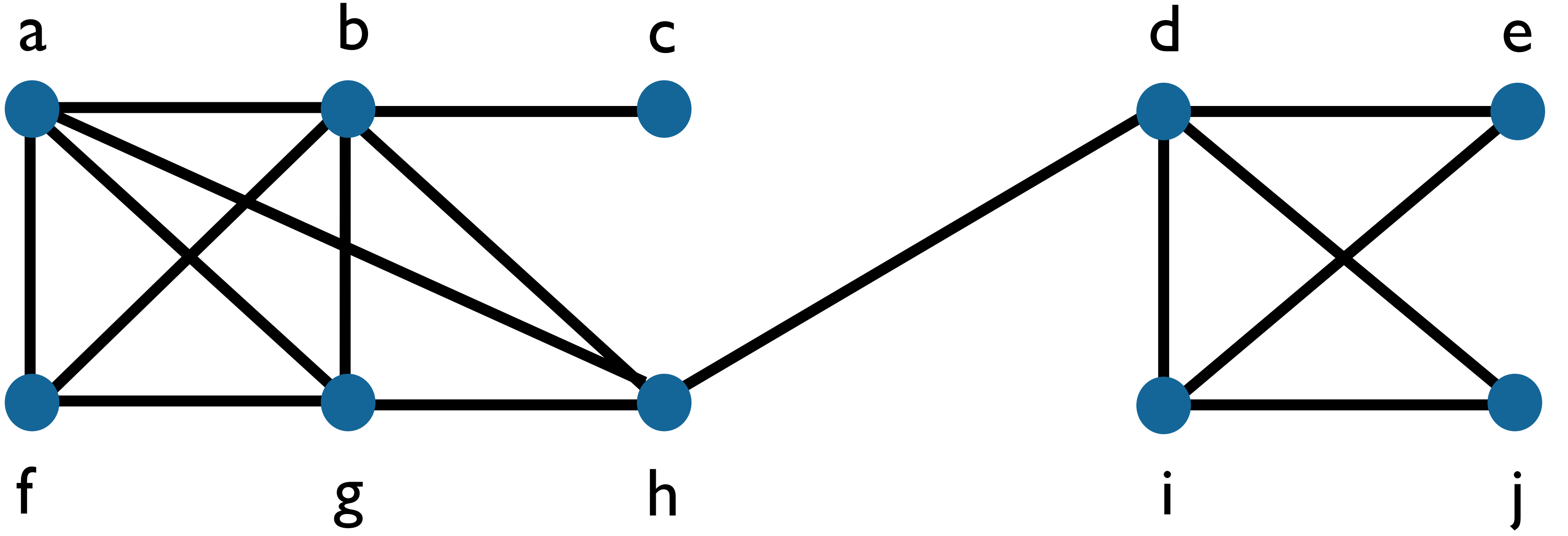
(Boob et. al, 2019)



a	b	c	d	e	f	g	h	i	j	
4	5	1	4	2	3	4	4	3	2	degree
X	X	X	X	X	X	X	1	X	X	curr. degree
3	2	1	1	2	3	1	0	2	1	loads

iterative peeling

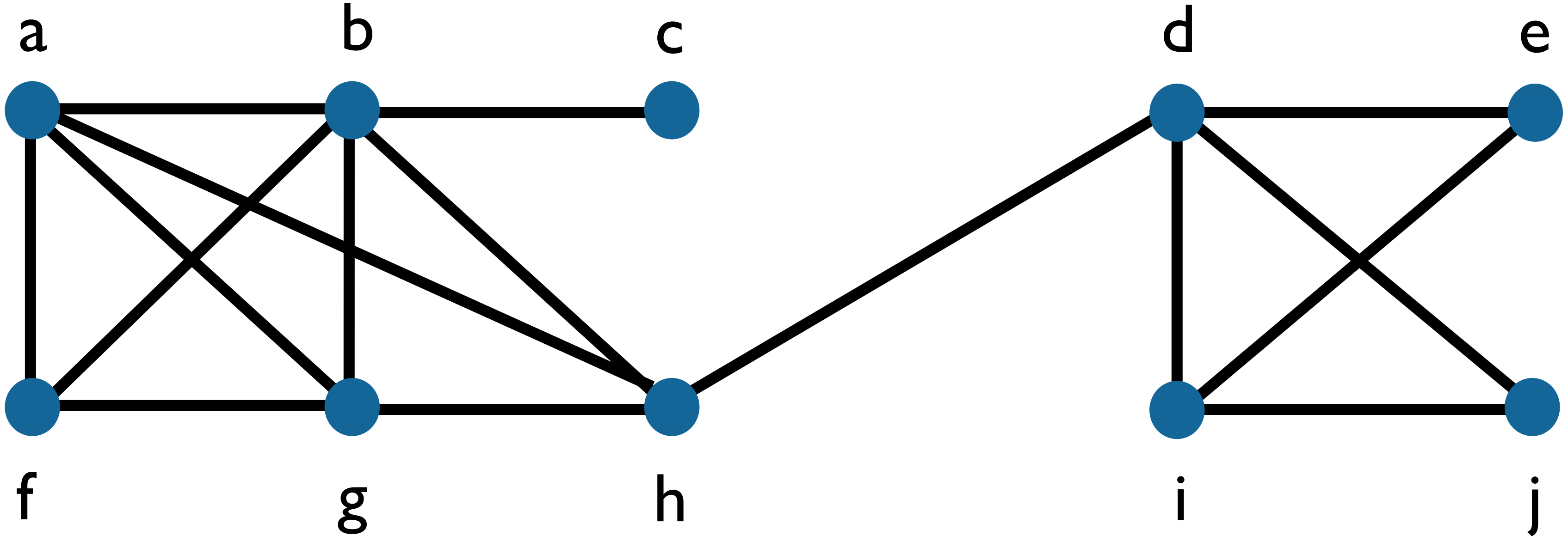
(Boob et. al, 2019)



a	b	c	d	e	f	g	h	i	j	
3	2	1	1	2	3	1	0	2	1	load
4	5	1	4	2	3	4	4	3	2	degree
7	7	2	5	4	6	5	4	5	3	curr load + deg
										load update

iterative peeling

(Boob et. al, 2019)

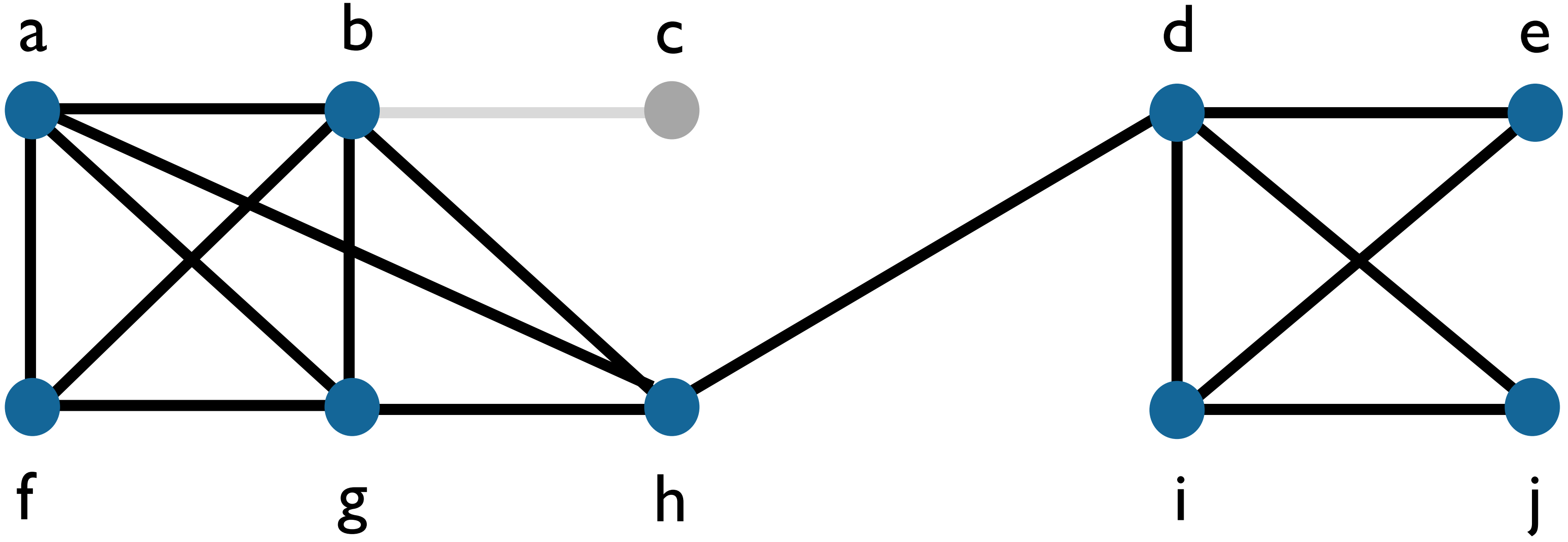


We obtain a new peeling order that changes with further iterations and eventually stabilizes.

a	b	c	d	e	f	g	h	i	j	
3	2	1	1	2	3	1	0	2	1	load
4	5	1	4	2	3	4	4	3	2	degree
7	7	2	5	4	6	5	4	5	3	curr load + deg
										load update

iterative peeling

(Boob et. al, 2019)



We obtain a new peeling order that changes with further iterations and eventually stabilizes.

a	b	c	d	e	f	g	h	i	j	
3	2	1	1	2	3	1	0	2	1	load
4	5	X	4	2	3	4	4	3	2	degree
7	7	1	5	4	6	5	4	5	3	curr load + deg
		1								load update

iterative peeling: summary

- Boob et. al experimentally showed that this seems to always eventually work!

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- Conjecture: this is a $(1-\epsilon)$–approximation for DSP with $(O\frac{1}{\epsilon^2})$ iterations required

iterative peeling: summary

- Boob et. al experimentally showed that this seems to always eventually work!
- Conjecture: this is a $(1-\epsilon)$–approximation for DSP with $(O\frac{1}{\epsilon^2})$ iterations required
- can we use this method to solve other problems?

Part 3:

The Densest Supermodular Set Problem (DSSP)

(iterative peeling for DSSP; convergence)

set functions

A set function assigns values to subsets of a set. We call the overall set we are working with the ground set.

set functions

A set function assigns values to subsets of a set. We call the overall set we are working with the ground set.

In other words, a set function is a function from the powerset of S to the real numbers.

$$f : 2^S \to \mathbb{R} \cup \{\pm\infty\}$$

set functions (subtypes)

Let V be a ground set, and let $f : 2^V \to \mathbb{R}$.

f is:

set functions (subtypes)

Let V be a ground set, and let $f : 2^V \to \mathbb{R}$.

f is:

- normalized if $f(\emptyset) = 0$

set functions (subtypes)

Let V be a ground set, and let $f : 2^V \to \mathbb{R}$.

f is:

- normalized if $f(\emptyset) = 0$
- monotone increasing if
  $A \subset B \implies f(A) < f(B)$

set functions (subtypes)

Let V be a ground set, and let $f : 2^V \to \mathbb{R}$.

f is:

- normalized if $f(\emptyset) = 0$
- monotone decreasing if
  $A \subset B \implies f(A) > f(B)$

set functions (subtypes)

Let V be a ground set, and let $f : 2^V \to \mathbb{R}$

f is:

- additive if $f(A \dot\cup B) = f(A) + f(B)$

set functions (subtypes)

Let V be a ground set, and let $f : 2^V \to \mathbb{R}$.

f is:

- additive if $f(A \dot\cup B) = f(A) + f(B)$
- subadditive if $f(A \dot\cup B) \leq f(A) + f(B)$

set functions (subtypes)

Let V be a ground set, and let $f : 2^V \to \mathbb{R}$.

f is:

- additive if $f(A \dot\cup B) = f(A) + f(B)$
- superadditive if
  $f(A \dot\cup B) \geq f(A) + f(B)$

set functions (marginal values)

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The marginal value of adding a new element to a set is the gain or loss incurred by adding that element to the set. Formally,

$$f(v|S) = f(S \cup \{v\}) - f(S), \;\; S \subsetneq V, v \notin S$$

submodularity, formally

A submodular function is a real-valued set function characterized by diminishing returns:

$$f(A \cup \{v\}) - f(A) \geq f(B \cup \{v\}) - f(B)$$

$$A \subsetneq B, \;\; v \notin B.$$

supermodularity, formally

A supermodular function is a real-valued set function characterized by increasing returns:

$$f(A \cup \{v\}) - f(A) \leq f(B \cup \{v\}) - f(B)$$

$$A \subsetneq B, \; v \notin B.$$

modularity, formally

A modular function is both submodular and supermodular.

$$f(A \cup \{v\}) - f(A) = f(B \cup \{v\}) - f(B)$$

$$A \subsetneq B, \;\; v \notin B.$$

Notice that modular functions are additive!

rewritten using marginal values…

submodular:

$$f(v|A) \geq f(v|B), \;\; A \subsetneq B, \;\; v \notin B.$$

rewritten using marginal values…

submodular:

$$f(v|A) \geq f(v|B), \; A \subsetneq B, \; v \notin B.$$

supermodular:

$$f(v|A) \leq f(v|B), \; A \subsetneq B, \; v \notin B.$$

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$$f(v|A) \geq f(v|B),\ A \subsetneq B,\ v \notin B.$$

supermodular:

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modular:

$$f(v|A) = f(v|B),\ A \subsetneq B,\ v \notin B.$$

The densest subgraph problem (DSP) is a special case of the densest supermodular set problem (DSSP).

(Charikar, Quanrud, Torres, 2022)

The DSSP, formally.

Given a non-negative supermodular function $f : 2^V \to \mathbb{R}^{\geq 0}$, let $S \subseteq V$. Then,

$$density(S) = \frac{f(S)}{|S|}$$

The DSSP, formally.

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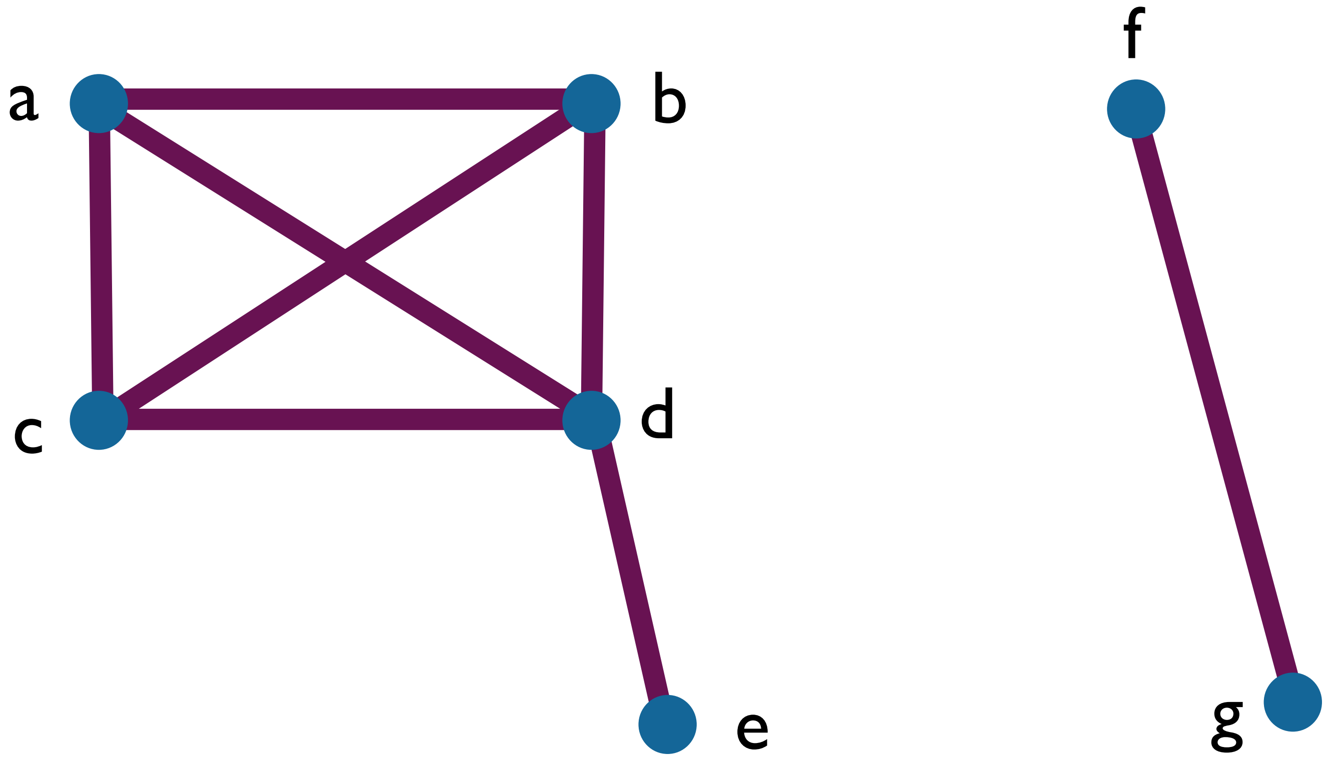
$$density(S) = \frac{f(S)}{|S|}$$

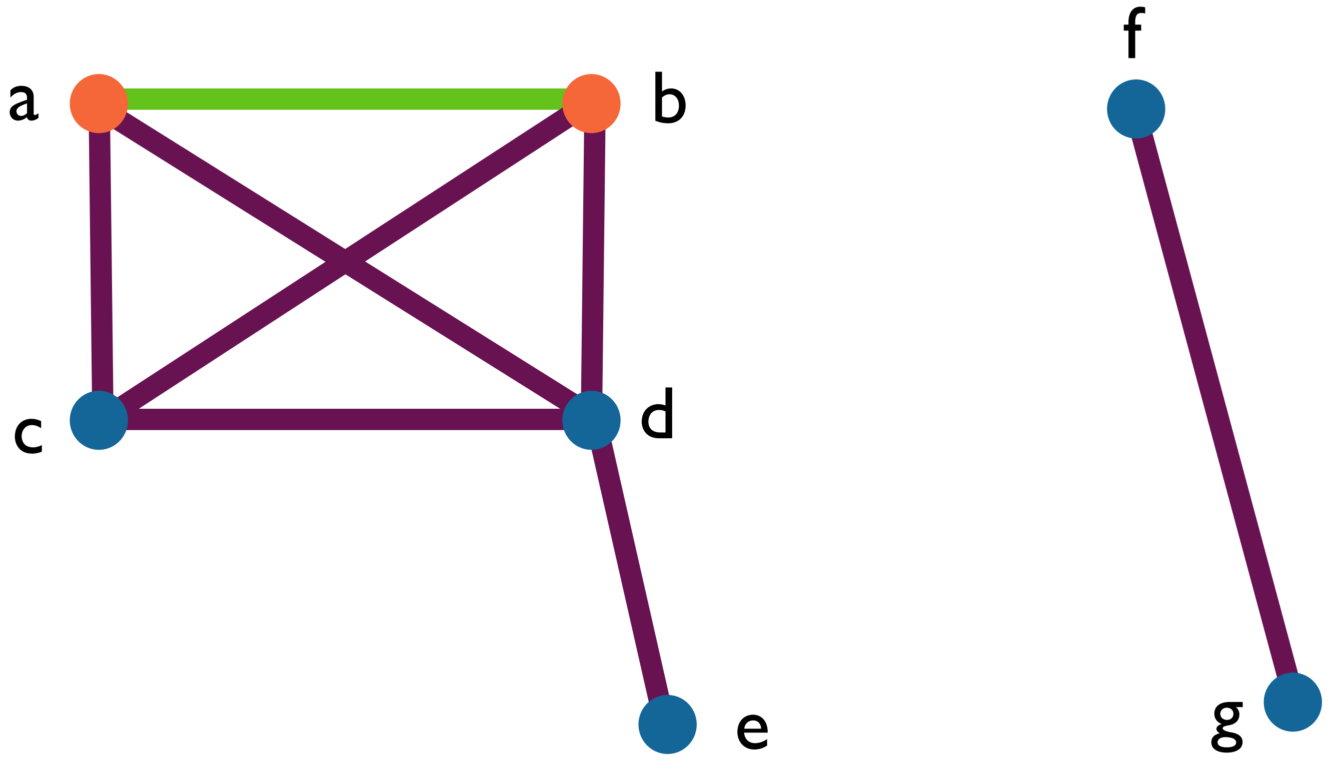
and we want to find the subset with the optimal density, λ^*:

$$\lambda^* = \max_{S \subseteq V} \frac{f(S)}{|S|}$$

|E(S)| is supermodular!

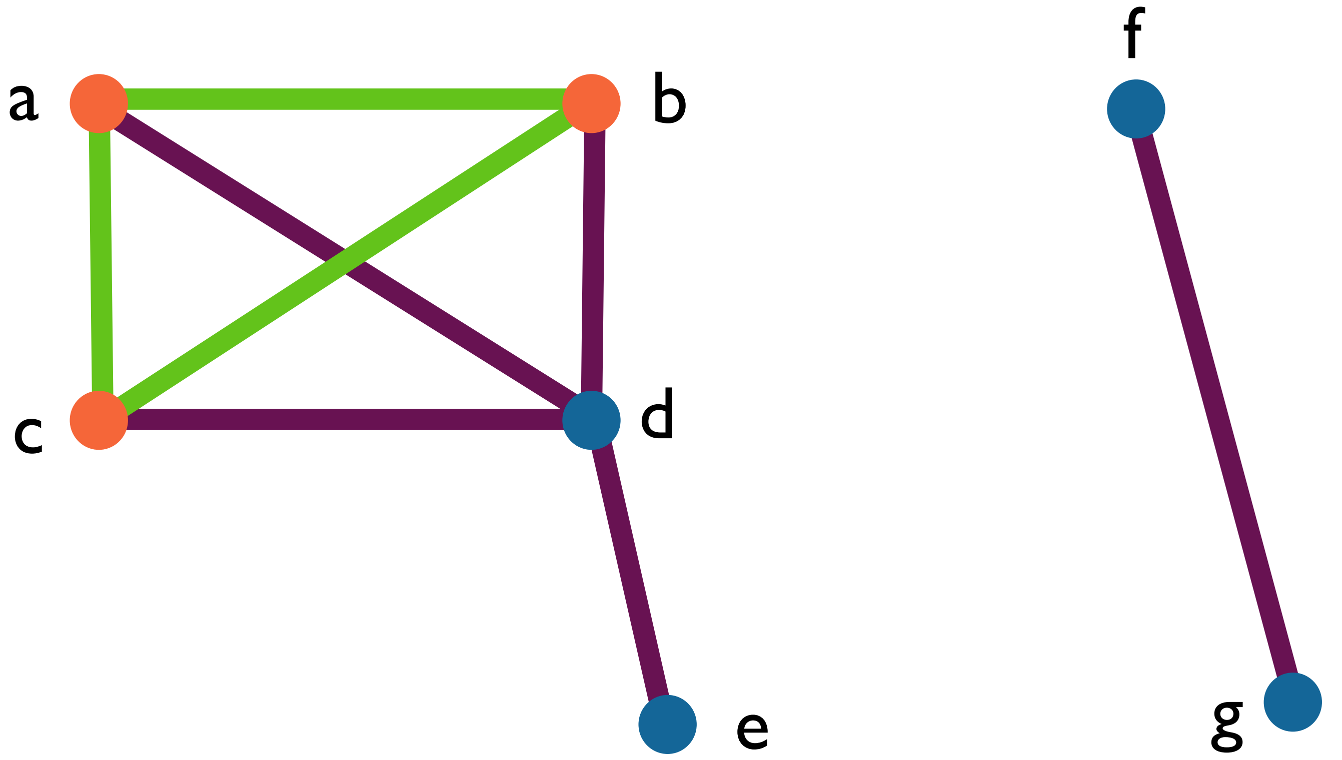


|E(S)| is supermodular!



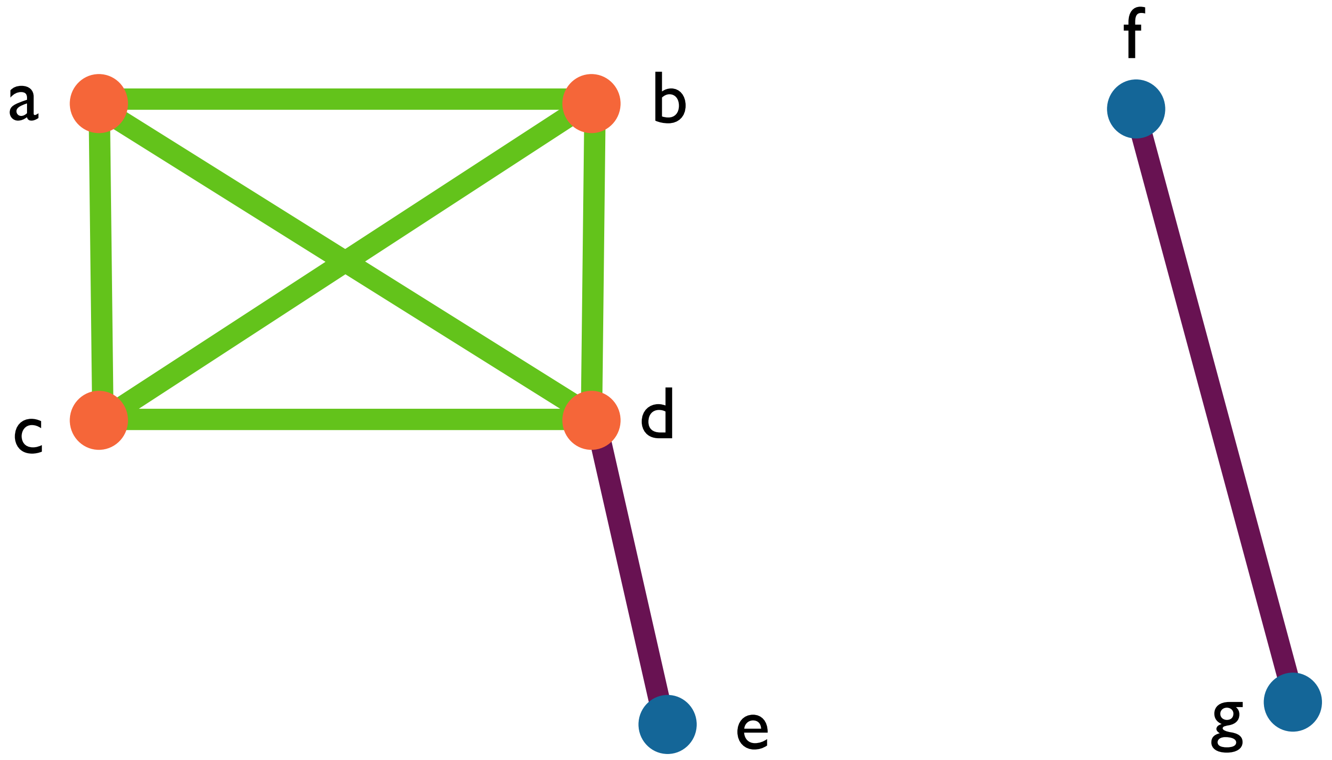
If we take S = {a, b}, then |E(S)| = 1.

|E(S)| is supermodular!



If we take T = {a, b, c}, then |E(T)| = 3. So f(c|S) = 2.

|E(S)| is supermodular!



If we take U = {a, b, c, d}, then |E(U)| = 6. So f(d|U) = 3!

SuperGreedy++, in general

- In general, we can replace the concept of current degree with the marginal value of v to the current set

(Charikar, Quanrud, Torres, 2022)

SuperGreedy++, in general

- In general, we can replace the concept of current degree with the marginal value of v to the current set
- Everything else about iterative peeling remains exactly the same.

(Charikar, Quanrud, Torres, 2022)

SuperGreedy++: summary

- This is a $(1-\epsilon)$– approximation for the DSSP with $O(1 - \frac{1}{\epsilon^2} \cdot \frac{\delta}{\lambda^*} \log n)$ iterations required

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SuperGreedy++: summary

- This is a $(1-\epsilon)$– approximation for the DSSP with $O(1 - \frac{1}{\epsilon^2} \cdot \frac{\delta}{\lambda^*} \log n)$ iterations required
- Therefore this is also a $(1-\epsilon)$– approximation for the DSP
- Iterative peeling works for any set with a supermodular function

(Charikar, Quanrud, Torres, 2022)

Main References

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