# Minimum-Cost Paths for Electric Cars

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# Introduction

Given

- an electric car with battery $b$ and capacity $B$
- a directed weighted graph $G = (V, A, c, r)$
  - $V$ be the set of vertices (junctions), $|V| = n$
  - $A$ be the set of edges (roads), $|A| = m$
  - $c : A \to \mathbb{R}$ is the cost of traversing an edge
  - $r : V \to \mathbb{R}^+ \cup \{\infty\}$ is the cost to charge at a vertex

# Introduction

- A car can traverse a road $(u, v) \in A$ only if it is in $u$ and the current charge $b \geq c(u, v)$.
- After traversing $(u, v)$, the car reaches $v$ with a charge of

$$\min\{b - c(u, v), B\}\,.$$

# Introduction

- To travel from $s \in V$ to $t \in V$, the car needs to choose a *travel plan* composed of a directed path $P$ from $s$ to $t$.
- We assume the car starts at $s$ with $b = 0$.
- The *cost* of the plan is the total charging cost along the selected path.
- Let $\rho_B(s, t)$ be the minimum cost of a travel plan from $s$ to $t$ with a max battery capacity of $B$.

# No Recharging Allowed

- $\alpha_{B,a}(s,t)$ is the Maximum Final Charge (MFC) the car can reach $t$ with starting from $s$ with charge $0 \leq a \leq B$.
  - If it is not possible, $\alpha_{B,a}(s,t) = -\infty$.
- $\beta_{B,b}(s,t)$ is the Minimum Initial Charge (MIC) needed to go from $s$ to $t$ with a charge of at least $b$ remaining.
  - If it is not possible, $\beta_{B,b}(s,t) = \infty$.

# Reduction

We reduce $\rho_B(s,t)$ to $\alpha_{B,a}(s,t)$ and $\beta_{B,b}(s,t)$, for $a, b \in \{0, B\}$, and to the solution of a standard all-pairs shortest paths problem.

# Travel Plans

- Let $u^a$, where $u \in V$ and $a \in [0, B]$, be the *state* of the car at $u$ with a charge of $a$.
- A *travel plan* is a sequence of states such that consecutive states correspond to either traversing an arc or recharging the battery.

# Travel Plans

A travel plan $P$ from $s$ to $t$ is the sequence

$$P = u_0^{a_0} u_1^{a_1} \dots u_l^{a_l}$$

where $u_0 = s$, $a_0 = 0$, and $u_l = t$.

- if $a_i \geq c(u_i, u_{i+1})$ and $a_{i+1} = \min\{a_i - c(u_i, u_{i+1}), B\}$
  - $u_i u_{i+1}$ is traversing an arc
- if $u_i = u_{i+1}$ and $a_i < a_{i+1}$
  - $u_i u_{i+1}$ is charging the battery

# Travel Plans

The cost of the plan is

$$\mathsf{cost}(P) = \sum_{i \in R} r(u_i)(a_{i+1} - a_i)$$

where $R = \{0 \leq i \leq l \mid u_i = u_{i+1}\}$ are the indices where rechargings take place.

# Structure of Optimal Plans

Segment $P$ such that no rechargings take place within each segment but instead occur between segments. Then

$$P = P_0 P_1 \dots P_k \,.$$

Each segment $P_i$ is of the form $P_i = x_i^{b_i} \dots x_{i+1}^{c_i}$ where $c_i < b_{i+1}$.

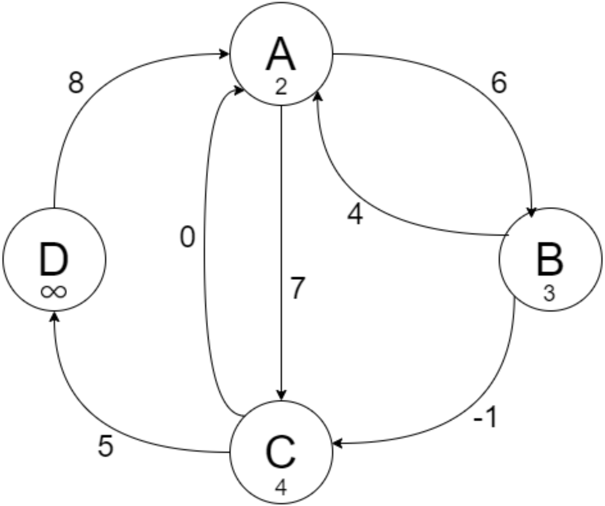
The whole plan $P$ is then

$$P = x_0^{b_0} \dots x_1^{c_1} | x_1^{b_1} \dots x_2^{c_2} | \dots | x_{k-1}^{b_{k-1}} \dots x_k^{c_{k-1}} | x_k^{b_k} \dots x_{k+1}^{c_k}$$

where each vertical line separates two segments and $x_1, \dots, x_k$ are vertices with recharging.

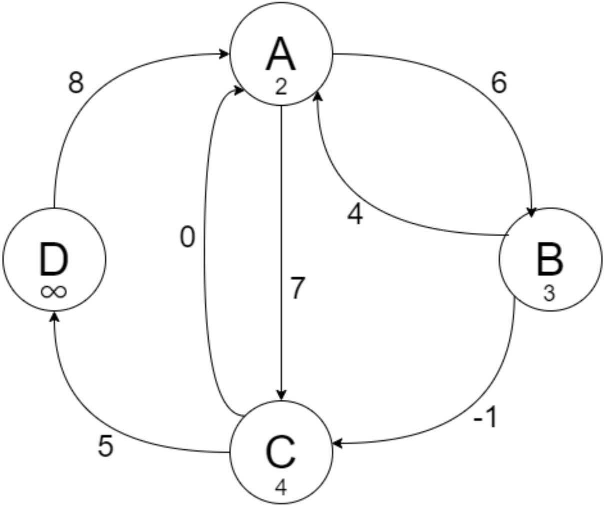
# Example

What is the minimum-cost plan from $A$ to $D$ when $B = 8$?



# Example (Continued)

$\rho_8(A, D) = 22,\ P = A^0|A^8B^2|B^4C^5D^0$



# Lemma 3.1

Lemma 3.1 states that the battery will always reach a state of either empty or full between the charging stations $x_i$ and $x_{i+1}$. That is, each segment of $P$ has at least one state with either an empty or full battery. Intuitively,

- If $r(x_i) > r(x_{i+1})$, it is optimal to fill the car just enough at $x_i$ to reach $x_{i+1}$ with an empty battery and recharge where it is cheaper.
- If $r(x_i) < r(x_{i+1})$, it is optimal to fully charge the battery at $x_i$ to minimize the need for recharging at $x_{i+1}$ where it is more expensive.

The actual vertices in which the battery is empty or full do not need to be $x_i$ or $x_{i+1}$, but it is more intuitive to think of them as such for simplicity.

# Lemma 3.2

For every travel plan, we are guaranteed intermediate states with an empty or full battery that have at most one recharging between them.

That is, there are states $y_0^{a_0}, \ldots, y_k^{a_k}$ such that $y_0^{a_0} = s^0$, $y_k^{a_k} = t^{a_k}$, and $a_i \in \{0, B\}$, and at most one recharging takes place between $y_i^{a_i}$ and $y_{i+1}^{a_{i+1}}$ on $P$, for $i = 0, \ldots, k-1$. [1]

[1] It's worth noting that $a_k$ does not have to be in {0,B}.

# Auxiliary Graph

Construct a complete graph with all states where the battery is either empty or full:

$$G^{0,B} = \left(V^0 \cup V^B, E\right) \qquad \text{where } V^a = \{u^a \mid u \in V\}\,.$$

Let $\ell_B(u^a, v^b)$ for $u, v \in V$ and $a, b \in \{0, B\}$, be the minimum cost of getting from $u$ to $v$ in the graph $G^{0,B}$, starting with an initial charge of $a$ at $u$ and reaching $v$ with a charge of at least $b$, with at most one recharge between them.

# Lemma 3.3

Lemma 3.3 states $\rho_B(s,t) = \delta_{\ell_B}(s^0, t^0)$, for every $s, t \in V$.

That is, the minimum-cost plan from $s$ to $t$ in the original graph $G$ is equal to the the cost of the shortest path from $s^0$ to $t^0$ in the graph $G^{0,B}$ with respect to $\ell_B$.

# Auxiliary Graph

Consider the cheapest path from $u^a$ to $v^b$ with a recharge at $x \in V$. We want to maximize $\alpha_{B,a}(u,x)$ and have exactly $\beta_{B,b}(x,v)$ charge at $x$ to reach $v$.

We need to buy $\max\{0, (\beta_{B,b}(x,v) - \alpha_{B,a}(u,x))\}$ charge at $x$.

# Auxiliary Graph

Considering all possible recharging vertices $x$, the cost $\ell_B(u^a, v^b)$ can be expressed as

$$\ell_B(u^a, v^b) = \min_{x \in V} \begin{cases} r(x) \cdot (\beta_{B,b}(x,v) - \alpha_{B,a}(u,x)) & \text{if } \alpha_{B,a}(u,x) < \beta_{B,b}(x,v)\,, \\ 0 & \text{otherwise.} \end{cases}$$

# Auxiliary Graph

To compute $\ell_B(u^a, v^b)$, we form a layered graph

$$(V^{0,B} \times U) \cup (U \times V^{0,B}),$$

where $U = \{x \in V \mid r(x) < \infty\}$ represents all charging stations.

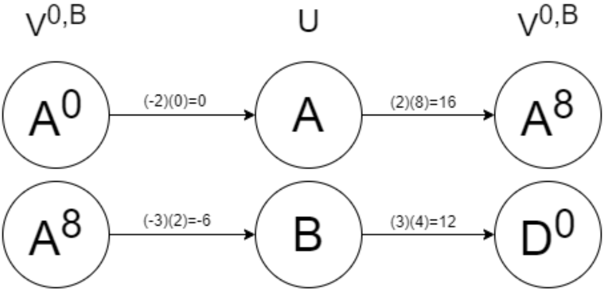
We define the weights of the edges in the graph as

$$w(u^a, x) = -r(x)\alpha_{B,a}(u, x) \quad \text{and} \quad w(x, v^b) = r(x)\beta_{B,b}(x, v)\,.$$

$\ell_B(u^a, v^b)$ is computed using a min-plus product in $O(n^3)$ time.

# Example (Continued)

$$\ell_8(A^0, A^8) = r(A) \cdot (\beta_{8,8}(A, A) - \alpha_{8,0}(A, A)) = 2 \cdot (8 - 0) = 16.$$
$$\ell_8(A^8, D^0) = r(B) \cdot (\beta_{8,0}(B, D) - \alpha_{8,8}(A, B)) = 3 \cdot (4 - 2) = 6.$$




$$\delta_{\ell_B}(A^0, D^0) = 22.$$

# The Reduction

Considering a graph with $n$ vertices and $m$ edges, let

- $MCP(m, n)$ be the problem of computing all-pairs Minimum-Cost Plans ($\rho_B(s, t)$ for every $s, t \in V$).
- $MFC(m, n)$ be the problem of computing all-pairs Maximum Final Charges ($\alpha_{B,a}(s, t)$ for every $s, t \in V$ for a given $0 \leq a \leq B$).
- $MIC(m, n)$ be the problem of computing all-pairs Minimum Initial Charges ($\beta_{B,b}(s, t)$ for every $s, t \in V$, for a given $0 \leq b \leq B$).
- $MPP(n, p)$ be the problem of computing a Min-Plus Product of an $(n \times p)$ matrix and a $(p \times n)$ matrix.
- $APSP(m, n)$ be the problem of computing the All-Pairs Shortest Paths ($\delta(s, t)$ for every $s, t \in V$).

# The Reduction

Theorem 3.1 states that $MCP(m,n)$ can be reduced to

- two instances of $MFC(m,n)$ and $MIC(m,n)$ each
- an instance of $MPP(2n,p)$, where $p \leq n$ is the number of charging stations
- an instance of $APSP(4n^2,2n)$

As shown earlier, the all-pairs shortest paths in $G^{0,B}$ is equal to the all-pairs minimum-cost plans in the original graph $G$.

# Algorithms

Combining the reduction of the previous section, and available algorithms for the MFC, MIC, MPP, and APSP, the all-pairs minimum-cost plans problem can be solved in the following times:

- $O\left(\frac{n^3}{2^{c\sqrt{\log n}}} + mn\right) = O(n^3)$ time for some $c > 0$ in a graph with no negative cycles
- $O(mn^2 + n^3 \log n)$ in a graph that may have negative cycles

# References

[1] Dani Dorfman et al. "Minimum-cost paths for electric cars". In: *Symposium on Simplicity in Algorithms (SOSA)*. SIAM. 2024, pp. 100–101.

[2] Dani Dorfman et al. "Optimal energetic paths for electric cars". In: *arXiv preprint arXiv:2305.19015* (2023).