Dimensionality Reduction

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Metric Space

Metric Space $\langle X, d \rangle$

Let X be a set of n-points and let d be a distance measure associated with pairs of elements in X.

We say that $\langle X, d \rangle$ is a finite metric space if the function d satisfies metric properties, i.e.

(a) $\forall x \in X$, $d(x, x) = 0$,
(b) $\forall x, y \in X, x \neq y, d(x, y) > 0$,
(c) $\forall x, y \in X$, $d(x, y) = d(y, x)$ (symmetry), and
(d) $\forall x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

Isometric embedding

Embeddings

Let $\langle X, d \rangle$ and $\langle X', d' \rangle$ be two metric spaces.

Embedding: A map $f : X \to X'$ is called an embedding.

Isometric embedding (i.e., distance preserving) if for all $x, y \in X$, $d(x, y) = d'(f(x), f(y))$.

3-useful distance measures between a pair of points $p = (p_1, \ldots, p_k)$ and $q = (q_1, \ldots, q_k)$ in $\Re^k$.

1. L_2-norm (Euclidean): $||p - q||_2 = \sqrt{\sum_{i=1}^{k} (p_i - q_i)^2}$
2. L_1-norm (Manhattan): $||p - q||_1 = \sum_{i=1}^{k} |p_i - q_i|$
3. L_∞-norm: $||p - q||_\infty = \max\{|p_1 - q_1|, \ldots, |p_k - q_k|\}$

Motivating Problem

Input: X=Set of n-points in k-dimensional space, where $n >> 2^k$
Output: A pair of points that maximize L_1-distance.

Let $p = (p_1, \ldots, p_k)$ and $q = (q_1, \ldots, q_k)$ be two points in $\Re^k$,
$||p - q||_1 = \sum_{i=1}^{k} |p_i - q_i|$.
For example, $||(3,5) - (2,7)||_1 = |3-2| + |5-7| = 3$.

Naive Solution: Compute distance between every pair of points and find the pair with largest distance
Total Time $= O(k\binom{n}{2}) = O(kn^2)$.

Next: An algorithm using isometric embedding of $L_1^k \to L_\infty^{2^k}$ running in $O(2^k n)$ time.

Isometric embedding $f : L_1^k \to L_\infty^{2^k}$

Let $x = (x_1, \ldots, x_k) \in X$

Note that $||x||_1 = \sum_{i=1}^{k} |x_i| = \sum_{i=1}^{k} \text{sign}(x_i) x_i = \text{sign}(x) \cdot x$, where sign$(x)$ is the ± 1 vector of length k denoting the sign of each coordinate of x.

Claim 1

For any ± 1 vector $y = (y_1, \ldots, y_k)$ of length k $||x||_1 = \text{sign(x)} \cdot x \geq y \cdot x$.
Moreover, $||x||_1 = \max\{y \cdot x | y \in \{-1, 1\}^k\}$.

An Illustration

Claim 1

For any ± 1 vector $y = (y_1, \ldots, y_k)$ of length k $||x||_1 =$ sign(x) $\cdot x \geq y \cdot x$. Moreover, $||x||_1 = \max\{y \cdot x | y \in \{-1, 1\}^k\}$.

For $x = (-2, -3, 4)$, $||x||_1 = |-2| + |-3| + |4| = (-1, -1, 1) \cdot (-2, -3, 4) = 9$

$y \cdot x$		
$(\mathbf{-1, -1, 1}) \cdot (\mathbf{-2, -3, 4})$	=	$\mathbf{9}$
$(-1, 1, 1) \cdot (-2, -3, 4)$	=	3
$(1, -1, 1) \cdot (-2, -3, 4)$	=	5
$(1, 1, 1) \cdot (-2, -3, 4)$	=	-1
$(-1, -1, -1) \cdot (-2, -3, 4)$	=	1
$(-1, 1, -1) \cdot (-2, -3, 4)$	=	-5
$(1, -1, -1) \cdot (-2, -3, 4)$	=	-3
$(1, 1, -1) \cdot (-2, -3, 4)$	=	-9

Isometric embedding $f : L_1^k \to L_\infty^{2^k}$ (contd.)

For each ± 1 vector y, define $f_y : X \to \Re$ by $f_y(x) = y \cdot x$

For example, $f_{(1,-1,1)}((-2,-3,4)) = (1,-1,1) \cdot (-2,-3,4) = 5$

Isometric Embedding

Define $f : X \to \Re^{2^k}$ to be the concatenation of f_y's for all possible 2^k $y's$.

For our example, $f(x) = (9, 3, 5, -1, 1, -5, -3, -9)$ corresponding to $2^3 = 8$ possible values for 3-dimensional vector y.

Let $x = (-2, -3, 4)$ and $x' = (2, 3, -2)$.

$||x - x'||_1 = |-2-2| + |-3-3| + |4-(-2)| = 16$

$f(x') = (-7, -1, -3, 3, -3, 3, 1, 7)$.

Observe
$||f(x) - f(x')||_\infty = \max_y\{|f_y(x) - f_y(x')|\} = \max(|9-(-7)|, |3-(-1)|, |5-(-3)|, |-1-3|, |1-(-3)|, |-5-3|, |-3-1|, |-9-7|) = 16 = ||x - x'||_1$

Isometric embedding $f : L_1^k \to L_\infty^{2^k}$ (contd.)

Isometric Embedding Lemma

Under the mapping $f : X \to \Re^{2^k}$ given by the concatenation of f_y's for all possible 2^k $y's$, where $f_y(x) = y \cdot x$, we have that for any two points $x, x' \in X$, $||f(x) - f(x')||_\infty = ||x - x'||_1$

Proof Sketch:

$$\begin{aligned} ||f(x) - f(x')||_\infty &= \max_y\{|f_y(x) - f_y(x')|\} \\ &= \max_y\{|y \cdot x - y \cdot x'|\} \\ &= \max_y\{|y \cdot (x - x')|\} \\ &= ||x - x'||_1, \end{aligned}$$

(by Claim 1 $||x||_1 = \max\{y \cdot x | y \in \{-1, 1\}^k\}$)

$\square$

Isometric embedding $f : L_1^k \to L_\infty^{2^k}$ (contd.)

In place of finding the furthest pair of points in X with respect to L_1 metric we have the following:

New Problem: Given n points in 2^k dimensional space X', find the furthest pair in X' with respect to L_∞ metric.

$$\begin{aligned} \max_{u,v \in X'} ||u - v||_\infty &= \max_{u,v \in X'} \max_{i=1}^{2^k} |u_i - v_i| \\ &= \max_{i=1}^{2^k} \max_{u,v \in X'} |u_i - v_i| \end{aligned}$$

Fix a coordinate, find the pair of points that maximize the difference with respect to that coordinate. Among all the coordinates, pick the one that maximizes the difference.

Furthest pair using L_∞ metric

Observe that $\max_{u,v \in X'} |u_i - v_i|$, for a fixed i, can be computed in $O(n)$ time

$\implies \max_{i=1}^{2^k} \max_{u,v \in X'} |u_i - v_i|$ can be computed in $O(2^k n)$ time.

Theorem

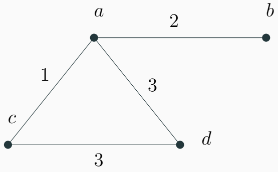
Given a set X of n points in $\Re^k$, by using the isometric embedding $f : L_1^k \to L_\infty^{2^k}$, we can compute the furthest pair of points in X with respect to L_1-metric by computing the furthest pair of points in the embedding with respect to L_∞-metric in $O(2^k n)$ time.

Universal Spaces

Universality of L_∞-metric space

Let $\langle X, d\rangle$ be any finite metric space, where $n = |X|$.
X can be isometrically embedded into L_∞-metric space of dimension n.

Proof Sketch: Let $X = \{x_1, \ldots, x_n\}$.
For each point $x \in X$, define $f(x) = (d(x, x_1), \ldots, d(x, x_n))$.



For example, let $X = \{a, b, c, d\}$, and we have

$$f(a) = (d(a,a), d(a,b), d(a,c), d(a,d)) = (0,2,1,2)$$
$$f(b) = (d(b,a), d(b,b), d(b,c), d(b,d)) = (2,0,3,5)$$
$$f(c) = (d(c,a), d(c,b), d(c,c), d(c,d)) = (1,3,0,3)$$
$$f(d) = (d(d,a), d(d,b), d(d,c), d(d,d)) = (3,5,3,0)$$

$$d(b,d) = ||f(b) - f(d)||_\infty = 5$$
$$d(a,d) = ||f(a) - f(d)||_\infty = 3$$

Universality of L_∞-metric (contd.)

Claim

For any pair of points $u, v \in X$, we have $d(u,v) = ||f(u) - f(v)||_\infty$

Proof:

$$\begin{aligned} ||f(u) - f(v)||_\infty &= \max_{x \in X} |d(u,x) - d(v,x)| \\ &\leq d(u,v) \text{ by triangle inequality} \end{aligned}$$

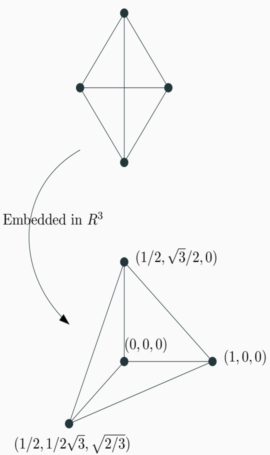
But, $\max_{x \in X} |d(u,x) - d(v,x)| \geq |d(u,u) - d(v,u)| = d(u,v)$
$\implies ||f(u) - f(v)||_\infty = d(u,v)$

□

$\implies$ the mapping of elements of $x \in X$ given by $f(x) = (d(x,x_1), \ldots, d(x,x_n))$ under L_∞-norm is universal.

Euclidean Metric

Input: Metric Space defined by K_4, C_4, and a star w.r.t. unweighted SP.
Question: Can one embed 4-points in Euclidean space (L_2) in any dimension isometrically?



Distortion

Distortion

Contraction: Is the maximum factor by which the distances shrink and it equals $\max\limits_{x,y \in X} \frac{d(x,y)}{d'(f(x),f(y))}$.

Expansion: Is the maximum factor by which the distances are stretched and it equals $\max\limits_{x,y \in X} \frac{d'(f(x),f(y))}{d(x,y)}$.

Distortion: of an embedding is the product of its expansion and contraction factor.

L_∞ Norm

Input: A metric space $\langle X, d\rangle$, where X is a set of n-points and let d satisfies the metric properties.
Output: An embedding of X in a $k = O(Dn^{\frac{2}{D}}\log n)$ dimensional space such that the distances gets distorted (actually contracted) by a factor of at most D under L_∞ norm.

We denote this embedding by the following notation:

$$\langle X, d\rangle \overset{D}{\hookrightarrow} L_\infty^{k=O(Dn^{\frac{2}{D}}\log n)}$$

Note, when $D = O(\log n)$, we have

$$\langle X, d\rangle \overset{\log n}{\hookrightarrow} L_\infty^{k=O(\log^2 n)}$$

I.e. we can embed any metric space in $O(\log^2 n)$ dimensional L_∞-metric space and the distances are distorted by a factor of $O(\log n)$.

Let $x, y \in X$ and let $f(x), f(y)$ be their embedding in the k-dimensional space, respectively.

Property

The distances gets contracted by a factor of at most $D \geq 1$. Formally, $\max_{x,y\in X} \frac{d(x,y)}{||f(x)-f(y)||_{\infty}} \leq D$

Example: If $D = O(\log n)$, $k = O(\log^2 n)$, i.e. $\langle X, d\rangle \overset{O(\log n)}{\hookrightarrow} L_{\infty}^{O(\log^2 n)}$

Meaning: Any metric space $\langle X, d\rangle$ can be embedded in a $O(\log^2 n)$-dimensional space and the distances may distort (contract) by a factor of at most $O(\log n)$.

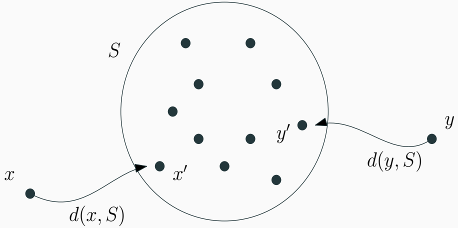
Space Saving Embedding: $\langle X, d\rangle$, where $n = |X|$, may require $O(n^2)$ space to capture distances between points. Whereas, in the mapped k-dimensional space, we only need to store $k = O(\log^2 n)$ coordinates for each point, thus requiring a total of $O(n \log^2 n)$ space.

Proof of $\langle X, d\rangle \overset{D}{\hookrightarrow} L_\infty^{k=O(Dn^{\frac{2}{D}}\log n)}$

Constructive proof via a randomized algorithm.

Definition

Let $S \subseteq X$. For $x \in X$, define the distance of x to the set S as $d(x, S) = \min_{z \in S} d(x, z)$



Claim 1

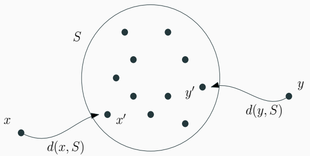
Let $x, y \in X$. For all $S \subseteq X$, $|d(x, S) - d(y, S)| \leq d(x, y)$.

Proof of Claim 1

Claim 1

Let $x, y \in X$. For all $S \subseteq X$, $|d(x, S) - d(y, S)| \leq d(x, y)$.

Proof:



Let $|d(x, S) - d(y, S)| = |d(x, x') - d(y, y')|$.

If $d(x, x') \geq d(y, y')$
$d(x, x') - d(y, y') \leq d(x, y') - d(y', y) \leq d(x, y)$ (by triangle inequality)
else $d(y, y') - d(x, x') \leq d(y, x') - d(x, x') \leq d(x, y)$.

Thus, $|d(x, S) - d(y, S)| = |d(x, x') - d(y, y')| \leq d(x, y)$.

□

$\Longrightarrow$ Distance to a subset amounts to contraction.

Proof Contd.

Definition

(**Mapping**) Let $x \in X$. Let $S_1, S_2, \cdots, S_k \subseteq X$. The mapping f maps x to the point

$$f(x) = \{d(x, S_1), d(x, S_2), \cdots, d(x, S_k)\}.$$

Claim 2

Let $S_1, S_2, \cdots, S_k \subseteq X$. For any pair of points $x, y \in X$, $||f(x) - f(y)||_\infty \leq d(x, y)$.

Proof: Follows from Claim 1, as for each $1 \leq i \leq k$, $|d(x, S_i) - d(y, S_i)| \leq d(x, y)$.

$\square$

Randomized Algorithm

Input: Metric space $\langle X, d \rangle$ and an integer parameter D.
Output: A set of $O(Dm)$ subsets of X.

1. $p \leftarrow \min(\frac{1}{2}, n^{-\frac{2}{D}})$
2. $m \leftarrow O(n^{\frac{2}{D}} \log n)$
3. For $j \leftarrow 1$ to $\lceil \frac{D}{2} \rceil$ and
   For $i \leftarrow 1$ to m:
   Choose set S_{ij} by sampling each element of X independently with probability p^j
4. For each $x \in X$ return $f(x) = [d(x, S_{11}), \cdots d(x, S_{m1}),$
   $d(x, S_{12}), \cdots, d(x, S_{m2}), \cdots\ d(x, S_{1\lceil \frac{D}{2} \rceil}), \cdots, d(x, S_{m\lceil \frac{D}{2} \rceil})]$

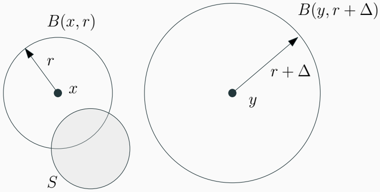
Intuition

- Each point $x \in X$ is embedded in $k = O(Dm)$ dimensional space via the mapping $f(x)$.
- By Claim 2, for any pair of points $x, y \in X$, $||f(x) - f(y)||_\infty \leq d(x, y)$, i.e. the distance shrinks.
- Fix a pair of points $x, y \in X$. We will prove a key lemma that states the following: *There exists an index $j \in \{1, \cdots, \lceil \frac{D}{2} \rceil\}$ such that if S_{ij} is as chosen in the Algorithm, than $Pr\left[||f(x) - f(y)||_\infty \geq \frac{d(x,y)}{D}\right] \geq \frac{p}{12}$.*
  In other words, under the L_∞-norm in the k-dimensional space, the distance doesn't shrink a lot!
- For index j we have m trials. So the probability that the above statement doesn't hold for all the m trials is $\leq (1 - \frac{p}{12})^m \leq e^{-\frac{pm}{12}} \leq \frac{1}{n^2}$. This follows from the choice of p and m as $p \leftarrow \min(\frac{1}{2}, n^{-\frac{2}{D}})$ and $m \leftarrow O(n^{\frac{2}{D}} \log n)$.
- We will apply the union bound to show that the above statement holds for all pairs of points with probability at least $1/2$.

An Observation

Observation 1

Let x, y be two distinct points of X. Let $B(x, r)$ be the set of points of X that are within a distance of r from x (think of $B(x, r)$ as a ball of radius r centred at x). Similarly, let $B(y, r + \Delta)$ be the set of points of X that are within a distance of $r + \Delta$ from y. Consider a subset $S \subset X$ such that $S \cap B(x, r) \neq \emptyset$ and $S \cap B(y, r + \Delta) = \emptyset$. Then $|d(x, S) - d(y, S)| \geq \Delta$.



Proof: $d(x, S) \leq r$ as $S \cap B(x, r) \neq \emptyset$

$d(y, S) \geq r + \Delta$ as $S \cap B(y, r + \Delta) = \emptyset$

$\implies |d(x, S) - d(y, S)| \geq \Delta$

$\square$

Ball Properties

Let $x, y \in X$. Set $\Delta = \frac{d(x,y)}{D}$.

Balls centred at x and y

For $i = 0, \cdots, \lceil \frac{D}{2} \rceil$, define balls of radius $i\Delta$ as follows.

Let $B_0 = \{x\}$.

B_1 be the ball of radius Δ centred at y.

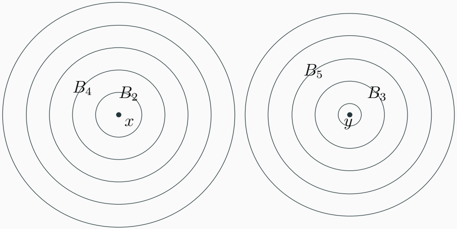
B_2 is the ball of radius 2Δ centred at x.

B_3 is the ball of radius 3Δ centred at y.

B_4 is the ball of radius 4Δ centred at x .

$\ldots$

$\ldots$



Properties of Balls

Property I

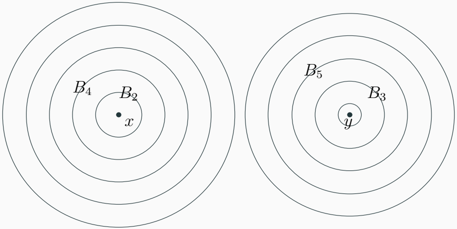
No balls centred at x overlaps with any of the balls centred at y.

Proof: Furthest point balls centred at x can reach is at distance $\leq \lceil \frac{D}{2} \rceil \Delta$.

Similarly, furthest point balls centred at y can reach is at distance $\leq (\lceil \frac{D}{2} \rceil - 1)\Delta$.

But $\lceil \frac{D}{2} \rceil \Delta + (\lceil \frac{D}{2} \rceil - 1)\Delta = 2\lceil \frac{D}{2} \rceil \Delta - \Delta < d(x, y)$, as $\Delta = \frac{d(x,y)}{D}$

□



Ball Properties (contd.)

For even (odd) i, let $|B_i|$ denote the number of points of X that are within a distance of at most $i\Delta$ from x (respectively, y).

Property II

There is an index $t \in \{0, \cdots, \lceil \frac{D}{2} \rceil - 1\}$, such that $|B_t| \geq n^{\frac{2t}{D}}$ and $|B_{t+1}| \leq n^{\frac{2(t+1)}{D}}$

Proof: Proof by contradiction.

$t = 0$: Since $|B_0| = 1 \implies |B_1| > n^{\frac{2}{D}}$

$t = 1$: If $|B_1| > n^{\frac{2}{D}} \implies |B_2| > n^{\frac{4}{D}}$

$t = 2$: If $|B_2| > n^{\frac{4}{D}} \implies |B_3| > n^{\frac{6}{D}}$

$\cdots$

$t = \lceil \frac{D}{2} \rceil - 1$: If $|B_t| > n^{\frac{2t}{D}} \implies |B_{\lceil \frac{D}{2} \rceil}| > n^{\frac{2\lceil \frac{D}{2} \rceil}{D}} \geq n$

But no ball can contain more than $|X| = n$ points. A contradiction.

□

Ball Properties (contd.)

Let t be the index such that $|B_t| \geq n^{\frac{2t}{D}}$ and $|B_{t+1}| \leq n^{\frac{2(t+1)}{D}}$
Consider when $j = t + 1$ in the Algorithm.

Property III

The set S_{ij} chosen by the algorithm has non-empty intersection with B_t with probability at least $p/3$, and it avoids B_{t+1} with probability at least $1/4$.

Define two events:
Event E_1: $S_{ij} \cap B_t \neq \emptyset$.
Event E_2: $S_{ij} \cap B_{t+1} = \emptyset$.
We will show that $Pr(E_1) \geq p/3$ and $Pr(E_2) \geq 1/4$.

By Property I, the balls B_t and B_{t+1} are disjoint.
Thus, $Pr(E_1 \wedge E_2) = Pr(E_1)Pr(E_2)$.
$\implies Pr(E_1 \wedge E_2) \geq \frac{p}{12}$.

Event E_1

$Pr(S_{ij} \cap B_t \neq \emptyset) \geq p/3$

Proof:

$$
\begin{aligned}
Pr(E_1) &= 1 - Pr(S_{ij} \cap B_t = \emptyset) \\
&= 1 - (1 - p^j)^{|B_t|} \text{ (No element of } B_t \text{ is chosen in } S_{ij}) \\
&= 1 - (1 - p^j)^{n^{\frac{2(j-1)}{D}}} \\
&\geq 1 - e^{-p^j n^{\frac{2(j-1)}{D}}} \\
&= 1 - e^{-p^j n^{\frac{2}{D} j} n^{-\frac{2}{D}}} \\
&= 1 - e^{-n^{-\frac{2}{D}}} \text{ (As } p = n^{-\frac{2}{D}}) \\
&= 1 - e^{-p}
\end{aligned}
$$

If $p < \frac{1}{2}$, $1 - e^{-p} \geq p/3$.

□

Event E_2

$Pr(S_{ij} \cap B_{t+1} = \emptyset) \geq 1/4$

Proof:

$$
\begin{aligned}
Pr(E_2) &= Pr(S_{ij} \cap B_{t+1} = \emptyset) \\
&= (1 - p^j)^{|B_{t+1}|} \\
&\geq (1 - p^j)^{n^{\frac{2j}{D}}} \\
&= (1 - p^j)^{\frac{1}{p^j}}
\end{aligned}
$$

If $p^j < \frac{1}{2}$, $(1 - p^j)^{\frac{1}{p^j}} \geq \frac{1}{4}$.

The function $(1 - p^j)^{\frac{1}{p^j}}$ achieves minimum at $p^j = 0$ or $p^j = \frac{1}{2}$, and in both the cases it is $\geq \frac{1}{4}$.

□

Lemma

Let x, y be two distinct points of X. There exists an index $j \in \{1, \cdots, \lceil \frac{D}{2} \rceil\}$ such that if S_{ij} is as chosen in the Algorithm, than $Pr\left[||f(x) - f(y)||_\infty \geq \frac{d(x,y)}{D}\right] \geq \frac{p}{12}$

1. $p \leftarrow \min(\frac{1}{2}, n^{-\frac{2}{D}})$
2. $m \leftarrow O(n^{\frac{2}{D}} \log n)$
3. For $j \leftarrow 1$ to $\lceil \frac{D}{2} \rceil$ and
   For $i \leftarrow 1$ to m:
   Choose set S_{ij} by sampling each element of X independently with probability p^j
4. For each $x \in X$ return $f(x) = [d(x, S_{11}), \cdots d(x, S_{m1}),$
   $d(x, S_{12}), \cdots, d(x, S_{m2}), \cdots \; d(x, S_{1\lceil \frac{D}{2} \rceil}), \cdots, d(x, S_{m\lceil \frac{D}{2} \rceil})]$

Proof of Key Lemma

Fix a pair of points $x, y \in X$. We know that $\Delta = \frac{d(x,y)}{D}$.

By Property II, there is a value of $t \in \{0, \ldots, \lceil \frac{D}{2} \rceil - 1\}$, such that $|B_t|$ is sufficiently large and $|B_{t+1}|$ is not too big. Choose $j = t + 1$.

By Property III, the probability that S_{ij} chosen by the algorithm overlaps with B_t and avoids B_{t+1} completely is at least $p/12$.

What is the probability that none of the m trials are good for that value of j?

$$\leq \left(1 - \frac{p}{12}\right)^m \leq e^{-\frac{pm}{12}} \leq \frac{1}{n^2}$$

as $p = \min(\frac{1}{2}, n^{-\frac{2}{D}})$ and $m = O(n^{\frac{2}{D}} \log n)$.

□

Main Theorem

$$\langle X, d\rangle \overset{D}{\hookrightarrow} L_{\infty}^{k=O(Dn^{\frac{2}{D}}\log n)}$$

Proof: For a fix pair of points $x, y \in X$, by the key lemma ,we have that there exists an index $j \in \{1, \cdots, \lceil \frac{D}{2} \rceil\}$ such that if S_{ij} is as chosen in the Algorithm, than $Pr\left[||f(x) - f(y)||_{\infty} \geq \frac{d(x,y)}{D}\right] \geq \frac{p}{12}$.

Moreover, as stated above, that this doesn't hold for all the m choices of S_{ij} is with probability at most $\frac{1}{n^2}$.

Since in all we have $\binom{n}{2}$ pairs of points in X, the probability of failure (for any pair) by the union bound is at most $\frac{1}{2}$.

$\implies$ probability of succeeding is $\geq \frac{1}{2}$

$\square$

Corollaries

Corollary 1

$\langle X, d \rangle \overset{\Theta(\log n)}{\hookrightarrow} L_{\infty}^{O(\log^2 n)}$

Proof: Set $D = \Theta(\log n)$, in the Theorem $\langle X, d \rangle \overset{D}{\hookrightarrow} L_{\infty}^{k=O(Dn^{\frac{2}{D}} \log n)}$ and we obtain $\langle X, d \rangle \overset{\Theta(\log n)}{\hookrightarrow} L_{\infty}^{O(\log^2 n)}$.

$\square$

Corollary 2

$\langle X, d\rangle \overset{\log^2 n}{\hookrightarrow} L_1^{O(\log^2 n)}$

Proof: Let $k = O(\log^2 n)$ be the dimension of embedding.
For a pair of points $x, y \in X$, we have $||f(x) - f(y)||_1 \leq kd(x, y)$ (it holds for each coordinate).

In the Theorem, for a pair $x, y \in X$, we know that there is at least one set which is good, i.e., with probability $\geq 1 - 1/n^2$, $||f(x) - f(y)||_\infty \geq \frac{d(x,y)}{\Theta(\log n)}$.

Extend the machinery in the Theorem to show that with high probability there are $\log n$ sets that are good by choosing slightly larger value for m (but still of order of $O(\log n)$). If this is the case, then
$||f(x) - f(y)||_1 \geq \log n \frac{d(x,y)}{\Theta(\log n)} = \Theta(d(x, y))$
Thus we have $\Theta(d(x, y)) \leq ||f(x) - f(y)||_1 \leq kd(x, y)$, and hence we have a mapping with distortion $O(\log^2 n)$.

$\square$

Corollary 3

$\langle X, d \rangle \overset{\log^{1.5} n}{\hookrightarrow} L_2^{O(\log^2 n)}$

Proof: Let $k = O(\log^2 n)$ be the dimension of embedding. Observe that for the same embedding as in Corollary 1, for a pair of points $x, y \in X$, we have

$$||f(x) - f(y)||_2 = \sqrt{\sum (d(x, S_{ij}) - d(y, S_{ij}))^2} \leq \sqrt{k} d(x, y)$$

We can show,

$$\begin{aligned} ||f(x) - f(y)||_2 &= \sqrt{\sum (d(x, S_{ij}) - d(y, S_{ij}))^2} \\ &\geq \sqrt{\log n \left(\frac{d(x, y)}{\Theta(\log n)} \right)^2} \\ &\geq \frac{d(x, y)}{\Theta(\sqrt{\log n})} \end{aligned}$$

This results in a total distortion of $O(\log^{1.5} n)$.

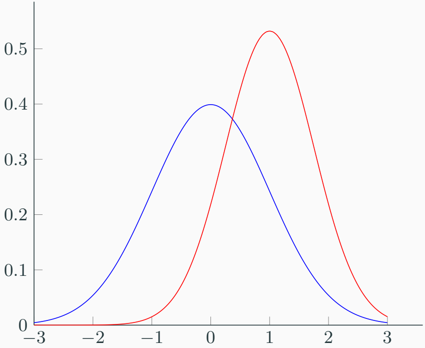
□

Normal Distribution

Normal Distribution

Random variable X has a *Normal Distribution* $\mathcal{N}(\mu, \sigma^2)$, with mean μ and standard deviation $\sigma > 0$, if its probability density function is of the form $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty$

Example: Plot of $\mathcal{N}(0, 1)$ and $\mathcal{N}(1, 0.75)$



Normal Distribution (contd.)

If X has a Normal distribution $\mathcal{N}(\mu, \sigma^2)$, than $aX + b$ has a Normal distribution $\mathcal{N}(a\mu + b, a^2\sigma^2)$, for constants a, b.

The distribution $\mathcal{N}(0, 1)$, with pdf $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, is referred to as the *standardized normal distribution*.

Sum of Normal Distributions

Let X and Y be independent r.v. with Normal distributions $\mathcal{N}(\mu_1, \sigma_1^2)$ and $\mathcal{N}(\mu_2, \sigma_2^2)$. Let r.v. $Z = X + Y$.
Z has a Normal distribution $\mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.
The sum of two independent Normal distributions is a Normal distribution.

L_2 Norm - Johnson-Lindenstrauss Theorem

Johnson-Lindenstrauss Theorem

Let V be a set of n points in d-dimensions. A mapping $f : \mathbb{R}^d \rightarrow \mathbb{R}^k$ can be computed, in randomized polynomial time, so that for all pairs of points $u, v \in V$,

$$(1-\epsilon)||u-v||^2 \leq ||f(u)-f(v)||^2 \leq (1+\epsilon)||u-v||^2,$$

where $0 < \epsilon < 1$ and n, d, and $k \geq 4(\frac{\epsilon^2}{2} - \frac{\epsilon^3}{3})^{-1} \ln n$ are positive integers.

Comments:

- The function f maps points of V to a $O(\frac{\ln n}{\epsilon^2})$-dimensional space from a d-dimensional space such that the distortion is within a factor of $1 \pm \epsilon$.
- $|| \cdot ||$ is with respect to Euclidean distance
- Function f is defined in terms of a matrix $A_{k \times d}$ with entries from Normal distribution $\mathcal{N}(0, \frac{1}{k})$.
- A point $v \in \Re^d$ is mapped to the point $v' = Av$. Note that $v' \in \Re^k$.

Matrix with entries from Normal distribution

- Let A be $k \times d$ dimensional matrix, where its entries are chosen independently from $\mathcal{N}(0, \frac{1}{k})$.

- Let x be a vector in R^d.

- Consider the k-dimensional vector Ax

- Next we show that the expected squared length of the vector $||Ax||^2$ is $||x||^2$.

Expected squared length

Lemma 1: $E[||Ax||^2] = ||x||^2$

Proof: Assume $z = Ax$, where $z = (z_1, \ldots, z_k) \in \Re^k$. We want to show that $E[||z||^2] = ||x||^2$.

Note that $||z||^2 = \sum_{i=1}^{k} z_i^2$.

Consider the first coordinate z_1 of z.

Note that $z_1 = \sum_{i=1}^{d} A_{1i} x_i$. What is the distribution of r.v. z_1?

Proof of $E[||Ax||^2] = ||x||^2$ (contd.)

1. Recall that if X has a Normal distribution $\mathcal{N}(0, \sigma^2)$, aX has a Normal distribution $\mathcal{N}(0, a^2\sigma^2)$, for a constant a. Moreover, the sum of two independent r.v. with Normal distributions $\mathcal{N}(0, \sigma_1^2)$ and $\mathcal{N}(0, \sigma_2^2)$ has a Normal distribution $\mathcal{N}(0, \sigma_1^2 + \sigma_2^2)$.
2. Since each A_{1i} is distributed independently by $\mathcal{N}(0, \frac{1}{k})$. The distribution of $z_1 = \sum_{i=1}^{d} A_{1i}x_i$ is the same as the sum of d independent Normal distributions (where each of them have an associated scalar x_i).
3. Thus, z_1 has $\mathcal{N}\left(0, \frac{\sum_{i=1}^{d} x_i^2}{k}\right) = \mathcal{N}\left(0, \frac{||x||^2}{k}\right)$ distribution.
4. Consider $||z||^2 = ||Ax||^2 = z_1^2 + \ldots + z_k^2$, where z_i has $\mathcal{N}(0, \frac{||x||^2}{k})$ distribution.
5. What is $E[||z||^2]$?

Proof of $E[||Ax||^2] = ||x||^2$

1. $E[||z||^2] = E[z_1^2 + \ldots + z_k^2] = kE[z_1^2]$
2. By definition: $Var[z_1] = E[z_1^2] - E[z_1]^2$.
   But z_1 has $\mathcal{N}\left(0, \frac{||x||^2}{k}\right)$ distribution
   $\implies Var[z_1] = \frac{||x||^2}{k}$ and $E[z_1] = 0$.
   $\implies E[z_1^2] = Var[z_1] = \frac{||x||^2}{k}$
3. Therefore, $E[||z||^2] = E[z_1^2 + \ldots + z_k^2] = kE[z_1^2] = ||x||^2$

$\square$

How good is the estimate $E[||Ax||^2] = ||x||^2$?

Is $E[||Ax||^2] = ||x||^2$ a good bound?

Estimate $Pr(||Ax||^2 \geq (1+\epsilon)||x||^2)$ and $Pr(||Ax||^2 \leq (1-\epsilon)||x||^2)$, for $\epsilon \in (0,1)$.

We know that $Pr(||Ax||^2 \geq (1+\epsilon)||x||^2) = Pr(\sum_{i=1}^{k} z_i^2 \geq (1+\epsilon)||x||^2)$, where z_i is a random variable with distribution $\mathcal{N}(0, \frac{||x||^2}{k})$.

Set $Y_i = \frac{\sqrt{k}}{||x||} z_i$.
Since z_i has distribution $\mathcal{N}(0, \frac{||x||^2}{k})$, Y_i has distribution $\mathcal{N}(0,1)$

In the expression $Pr(\sum_{i=1}^{k} z_i^2 \geq (1+\epsilon)||x||^2)$, divide by $\frac{||x||^2}{k}$, and we obtain $Pr(\sum_{i=1}^{k} Y_i^2 \geq (1+\epsilon)k)$.

New Problem

Estimate $Pr(\sum_{i=1}^{k} Y_i^2 \geq (1+\epsilon)k)$, where Y_i has a $\mathcal{N}(0,1)$ distribution.

Estimating $Pr(\sum_{i=1}^{k} Y_i^2)$

Lemma 2

1. $Pr(\sum_{i=1}^{k} Y_i^2 \geq (1+\epsilon)k) \leq e^{-\frac{k}{4}(\epsilon^2-\epsilon^3)}$
2. $Pr(\sum_{i=1}^{k} Y_i^2 \leq (1-\epsilon)k) \leq e^{-\frac{k}{4}(\epsilon^2-\epsilon^3)}$

Proof of 1:

$$
\begin{aligned}
Pr(\sum_{i=1}^{k} Y_i^2 \geq (1+\epsilon)k) &= Pr(e^{\lambda \sum_{i=1}^{k} Y_i^2} \geq e^{(1+\epsilon)\lambda k}) \text{ (for } \lambda > 0\text{)} \\
&\leq \frac{E\left[e^{\lambda \sum_{i=1}^{k} Y_i^2}\right]}{e^{(1+\epsilon)\lambda k}} \text{ (applying Markov's Inequality)} \\
&= \frac{E\left[e^{\lambda Y_1^2}\right]^k}{e^{(1+\epsilon)\lambda k}} \text{ (Independence of } Y_i\text{'s)}
\end{aligned}
$$

A useful identity

An Identity

Let X be a random variable distributed $\mathcal{N}(0,1)$ and $\lambda < \frac{1}{2}$ be a constant. Then, $E\left[e^{\lambda X^2}\right] = \frac{1}{\sqrt{1-2\lambda}}$

Proof: PDF of standard normal distribution is $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.

By definition, $E[H(x)] = \int\limits_{-\infty}^{+\infty} H(x) f(x) dx$

Thus, $E\left[e^{\lambda X^2}\right] = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{\lambda x^2} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{-(1-2\lambda)\frac{x^2}{2}} dx$

Substitute $y = x\sqrt{1-2\lambda}$, and we obtain

$$E\left[e^{\lambda X^2}\right] = \frac{1}{\sqrt{1-2\lambda}} \left[\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{-\frac{y^2}{2}} dy \right]$$

But, $\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{-\frac{y^2}{2}} dy = 1$, as this is the area under the Normal distribution curve.

$\square$

Proof of $Pr(\sum_{i=1}^{k} Y_i^2 \geq (1+\epsilon)k) \leq e^{-\frac{k}{4}(\epsilon^2-\epsilon^3)}$ (contd.)

We have

$Pr(\sum_{i=1}^{k} Y_i^2 \geq (1+\epsilon)k) \leq \frac{E\left[e^{\lambda Y_1^2}\right]^k}{e^{(1+\epsilon)\lambda k}} = e^{-(1+\epsilon)k\lambda}\left(\frac{1}{\sqrt{1-2\lambda}}\right)^k$ (using the identity)

Set $\lambda = \frac{\epsilon}{2(1+\epsilon)}$ and we have

$$
\begin{aligned}
Pr(\sum_{i=1}^{k} Y_i^2 \geq (1+\epsilon)k) &\leq e^{-(1+\epsilon)k\lambda}\left(\frac{1}{\sqrt{1-2\lambda}}\right)^k \\
&= e^{-\frac{\epsilon}{2}k}(1+\epsilon)^{\frac{k}{2}} \\
&= \left((1+\epsilon)e^{-\epsilon}\right)^{\frac{k}{2}} \\
&\leq e^{-\frac{k}{4}(\epsilon^2-\epsilon^3)} \text{ (as } 1+\epsilon \leq e^{\epsilon-\frac{\epsilon^2-\epsilon^3}{2}}\text{)}
\end{aligned}
$$

This finishes the proof of the 1st part of Lemma 2. The proof of 2nd part is similar and is left as an exercise.

□

Estimating $Pr(\sum_{i=1}^{k} Y_i^2)$

Corollary 1

If $k = c\frac{\ln n}{\epsilon^2}$, for some constant $c > 4$,

$Pr((1-\epsilon)k \leq \sum_{i=1}^{k} Y_i^2 \leq (1+\epsilon)k) \geq 1 - \frac{1}{n^3}$

Proof: From Lemma 2 we have that

$Pr(\sum_{i=1}^{k} Y_i^2 \geq (1+\epsilon)k) \leq e^{-\frac{k}{4}(\epsilon^2-\epsilon^3)}$ and $Pr(\sum_{i=1}^{k} Y_i^2 \leq (1-\epsilon)k) \leq e^{-\frac{k}{4}(\epsilon^2-\epsilon^3)}$.

Hence $Pr\left((\sum_{i=1}^{k} Y_i^2 \geq (1+\epsilon)k) \vee (\sum_{i=1}^{k} Y_i^2 \leq (1-\epsilon)k)\right) \leq 2e^{-\frac{k}{4}(\epsilon^2-\epsilon^3)}$ (by Union Bound)

Thus, $Pr((1-\epsilon)k \leq \sum_{i=1}^{k} Y_i^2 \leq (1+\epsilon)k) \geq 1 - 2e^{-\frac{k}{4}(\epsilon^2-\epsilon^3)}$

Substituting, $k = c\frac{\ln n}{\epsilon^2}$ we have that

$Pr((1-\epsilon)k \leq \sum_{i=1}^{k} Y_i^2 \leq (1+\epsilon)k) \geq 1 - \frac{1}{n^3}$ (bit sloppy computation)

$\square$

J-L Theorem

Let V be a set of n points in d-dimensions. A mapping $f : \mathbb{R}^d \rightarrow \mathbb{R}^k$ can be computed, in randomized polynomial time, so that for all pairs of points $u, v \in V$,

$$(1-\epsilon)||u-v||^2 \leq ||f(u)-f(v)||^2 \leq (1+\epsilon)||u-v||^2,$$

where $0 < \epsilon < 1$ and n, d, and $k \geq 4(\frac{\epsilon^2}{2} - \frac{\epsilon^3}{3})^{-1} \ln n$ are positive integers.

By choosing matrix $A_{k\times d}$ consisting of independent values from $\mathcal{N}(0, \frac{1}{k})$, we show that $\forall u, v \in V$
$Pr((1-\epsilon)||u-v||^2 \leq ||Au - Av||^2 \leq (1+\epsilon)||u-v||^2) \geq 1 - \frac{1}{n}$

Proof of J-L Theorem

Proof: By Corollary 1, we know that for any vector $x \in R^d$,
$Pr((1-\epsilon)||x||^2 \leq ||Ax||^2 \leq (1+\epsilon)||x||^2) \geq 1 - \frac{1}{n^3}$

Consider any pair of points $u, v \in V$. Set $x = u - v$. Then
$Pr((1-\epsilon)||u-v||^2 \leq ||A(u-v)||^2 \leq (1+\epsilon)||u-v||^2) \geq 1 - \frac{1}{n^3}$

There are in all $\binom{n}{2}$ pairs of points in V.
By union bound, we have that $\forall u, v \in V$
$Pr((1-\epsilon)||u-v||^2 \leq ||Au - Av||^2 \leq (1+\epsilon)||u-v||^2) \geq 1 - \frac{1}{n}$

$\square$

Comments

1. Choice of matrix A doesn't depend on points in V
2. What properties A needed to satisfy?
3. $E[||Ax||^2] = ||x||^2$
4. A is dense $\implies$ Av takes more computation time
5. Can we find sparse matrix A?
   Choose entries of A from $\{-1, 1, 0\}$ with probabilities $1/6$,$1/6$, and $2/3$, respectively and normalize.
6. $\ldots$

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