Twin-width

Yan Garito

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Context

Problem: Many problems are NP-hard on general graphs:

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- k-Independent graph;

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- The list goes on...

Solution: FPT algorithms

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- k-Independent graph;
- k-Dominating Set;
- k-Coloring;
- The list goes on...

Solution: FPT algorithms...but what parameter do we fix?

Some parameters to fix

- Treewidth (Bertele and Brioschi 1972; Halin 1976; Robertson and Seymour 1984)

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- **Twin-width** (Bonnet, Kim, et al. 2021)

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  - Unknown whether we can recognize graphs with clique-width k for constant k
- **Twin-width** (Bonnet, Kim, et al. 2021)
  - Useful even on dense graphs
  - Can mostly be computed (NP-hard still) : SAT solvers (Bergé, Bonnet, and Déprés 2022; Schidler and Szeider 2022)

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What is twin-width?

Definition (Trigraph)

Triple $G = (V, E, R)$:

- V vertex set;
- $E \subseteq V \times V$ **black** edge set
- $R \subseteq V \times V$ **red** edge set
- $E \cap R = \varnothing$

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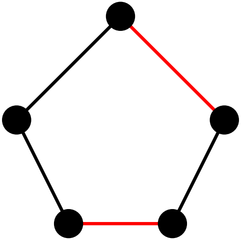


What is twin-width?

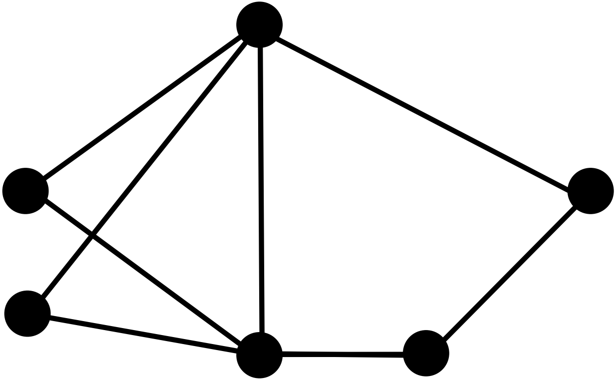
Definition (Contraction sequence)

G (tri)graph on n vertices. Contraction sequence $G = G_n, G_{n-1}, ..., G_2, G_1 = K_1$ if for all $i \in \{1, 2, ..., n-1\}$, G_{n-i} is obtained from G_{n-i+1} by merging two vertices u and v into a vertex uv as follows:

- If u and v both have a black edge with w, then uv has a black edge with w
- If neither u nor v have an edge with w, then uv has no edge with w
- Otherwise, uv has a red edge with w

What is twin-width?

uw and vw black $\Rightarrow (uv)w$ black
uw and vw missing $\Rightarrow (uv)w$ missing
otherwise $(uv)w$ red



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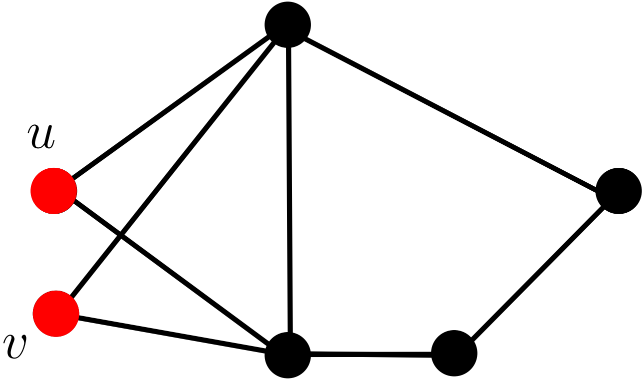

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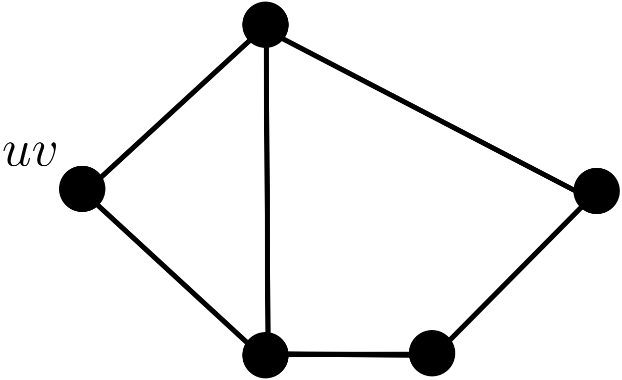

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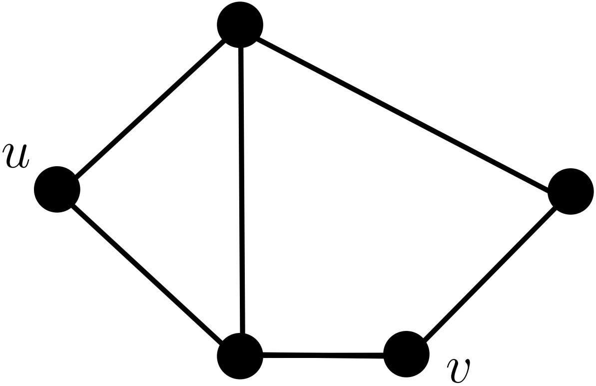

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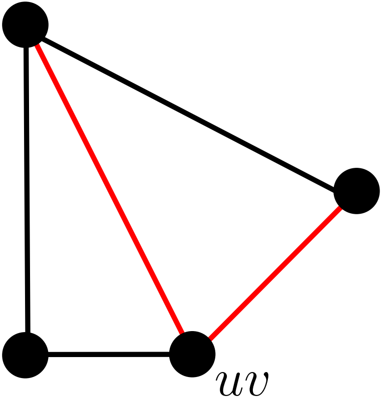

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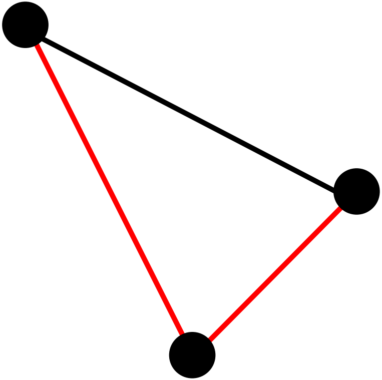

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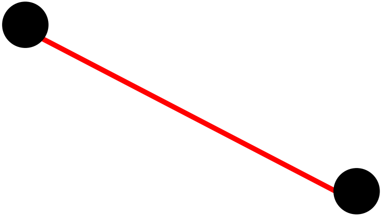

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What is twin-width?

Definition (Twin-width)

Let $G = (V, E)$. Denote by $\Delta_r(H)$ the maximum red degree of a trigraph H.

$$\text{tww}(G) = \min_{G_n,\ldots,G_1 \text{ contraction sequence of } G} \{ \max_{i=1,\ldots,n} \{\Delta_r(G_i)\}\}$$

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Definition ($u(G)$)

If $u \in G_i$, a trigraph in a contraction sequence, then $u(G) =$ the set of vertices of G that were contracted to form u

k-Independent Set

k-Independent Set

Definition (Independent Set)

An independent set I in a graph $G = (V, E)$ is a set such that $I \subseteq V$ and for any $u, v \in I, uv \notin E$. We denote by $\alpha(G)$ the size of the largest independent set of G

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Problem (k-Independent Set)

Input: *a graph* $G = (V, E)$*, an integer* $k \geq 1$
Output: *yes if there exists an independent set* $I \subseteq V$ *with* $|I| \geq k$ *and no otherwise*

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How does twin-width help?

k-Independent Set

Idea: Use the contraction sequence to progressively build up partial solutions (Bonnet, Geniet, et al. 2021)

k-Independent Set

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Definition (Partial solution)

A partial solution in the trigraph G_i is a pair (T, S):

- $T \subseteq V(G_i)$ is a connected subset of vertices in the red graph of G_i
- $S \subseteq V(G)$ is an independent set of G such that $S \subseteq \cup_{u \in T} u(G)$ and for every $u \in T, S \cap u(G) \neq \varnothing$

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Definition (Compatible partial solutions)

(T, S) and (T', S') are compatible if there is no edge (no matter the color) uu' in G_i such that $u \in T$ and $u' \in T'$

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Definition (Union of compatible partial solutions)

$(T, S) \cup (T', S') = (T \cup T', S \cup S')$

Remark

The union of two compatible partial solutions is not a partial solution

k-Independent Set

Definition (Realizable set)

A set $T \subseteq V(G_i)$ is realizable (in G_i) if there exists a set $S \subseteq V(G)$ such that (T, S) is a partial solution in G_i

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Definition (dec)

Let $X \subseteq V(G_i)$. The set $\text{dec}(X)$ is the set with one partial solution in G_i for each connected component of X in its red graph. If one connected component is not realizable, then $\text{dec}(X) = \text{None}$.
$\cup\,\text{dec}(X) = \text{None}$ if two of the partial solutions in X are not compatible or if $\text{dec}(X) = \text{None}$

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k-Independent Set

Algorithm: Input: $G = (V, E)$ a graph, $k \geq 1$ an integer and a contraction sequence $(G_n, ..., G_1)$ of G

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- Initialize the partial solutions $\mathcal{S}_n = \{(\{v\}, \{v\}), v \in V(G)\}$
- Let $u, v \in G_{i+1}$ be contracted to $z \in G_i$. Initialize $\mathcal{S}_i$ as all the solutions in $\mathcal{S}_{i+1}$ not intersecting $\{u, v\}$

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- Consider all realizable sets $T \ni z$ in G_i
    - For each such T, add the partial solution with largest S from among $\cup\text{dec}(T\backslash\{z\} \cup \{u, v\}), \cup\text{dec}(T\backslash\{z\} \cup \{u\})$ and $\text{dec}(T\backslash\{z\} \cup \{v\})$ to the set of partial solutions $\mathcal{S}_i$

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  - Before adding a partial solution to $\mathcal{S}_i$, we check if its S has size $\geq k$. If so, we return yes

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  - Before adding a partial solution to $\mathcal{S}_i$, we check if its S has size $\geq k$. If so, we return yes
- If we build $\mathcal{S}_1$ and still have not found a solution whose S has size $\geq k$, we return no

k-Independent Set

Theorem

This algorithm is correct

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k-Independent Set

Theorem

This algorithm is correct

- By induction, we show that for every realizable set $T \subseteq V(G_i)$, we have $(T, S) \in \mathcal{S}_i$ such that either $|S| = \alpha(G[\cup_{u \in T} u(G)])$ or $|S| \geq k$. We will always assume the former, as the algorithm terminates in the latter case

k-Independent Set

Theorem

This algorithm is correct

- IH: for every realizable set $T \subseteq V(G_i)$, we have $(T, S) \in \mathcal{S}_i$ with $|S| = \alpha(G[\cup_{u \in T} u(G)])$
- Let u, v be contracted into z from G_{i+1} to G_i

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- Let u, v be contracted into z from G_{i+1} to G_i
- Let T be realizable in G_i. There are two cases:

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- IH: for every realizable set $T \subseteq V(G_i)$, we have $(T, S) \in \mathcal{S}_i$ with $|S| = \alpha(G[\cup_{u \in T} u(G)])$
- Let u, v be contracted into z from G_{i+1} to G_i
- Let T be realizable in G_i. There are two cases:
  - $z \notin T$: Then we inherit S from $\mathcal{S}_{i+1}$ and by the induction hypothesis, it satisfies $|S| = \alpha(G[\cup_{u \in T} u(G)])$

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- Let u, v be contracted into z from G_{i+1} to G_i
- Let T be realizable in G_i. There are two cases:
  - $z \in T$:
    - Let S' be a maximum independent set of $G[\cup_{u \in T} u(G)]$.

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- Let T be realizable in G_i. There are two cases:
  - $z \in T$:
    - Let S' be a maximum independent set of $G[\cup_{u \in T} u(G)]$.
    - Build $\varnothing \neq I \subseteq \{u, v\}$ by adding u (resp. v) to I if $S' \cap u(G) \neq \varnothing$ (resp. $S' \cap v(G) \neq \varnothing$)

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    - Build $\varnothing \neq I \subseteq \{u, v\}$ by adding u (resp. v) to I if $S' \cap u(G) \neq \varnothing$ (resp. $S' \cap v(G) \neq \varnothing$)
    - By the existence of S', the connected components of $T \backslash \{z\} \cup I$ are all realizable
    - By the induction hypothesis, dec$(T \backslash \{z\} \cup I)$ contains optimal partial solutions, so its union is optimal

k-Independent Set

Theorem

This algorithm runs in time $O(k^2 d^{2k} n)$ where d is the maximum red degree of a graph in the contraction sequence

k-Independent Set

Theorem

This algorithm runs in time $O(k^2 d^{2k} n)$ where d is the maximum red degree of a graph in the contraction sequence

- Enumerating the T's takes time $O(d^{2k})$
- Checking compatibility of partial solutions takes time $O(k^2)$

k-Dominating Set

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Definition (Dominating Set)

A dominating set D in a graph $G = (V, E)$ is a set such that $D \subseteq V$ and for any $u \in V$, either $u \in D$ or there is $v \in D$ such that $uv \in E$

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A dominating set D in a graph $G = (V, E)$ is a set such that $D \subseteq V$ and for any $u \in V$, either $u \in D$ or there is $v \in D$ such that $uv \in E$

Problem (k-Dominating Set)

Input: *a graph* $G = (V, E)$*, an integer* $k \geq 1$
Output: *yes if there exists a dominating set* $D \subseteq V$ *with* $|D| \leq k$ *and no otherwise*

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k-Dominating Set

(Bonnet, Geniet, et al. 2021)

Definition (Profile of a partial solution)

A profile of a partial solution in G_i is a triple (T, D, M) of subsets of $V(G_i)$:

- T forms a connected subset in the red graph of G_i
- $D, M \subseteq T$
- $\cup_{x \in D} B_i^2(x) \subseteq T$ where $B_i^2(x)$ is the ball of radius 2 centered at x in the red graph of G_i

A profile such that $|D| \leq k$ is called a k-profile

k-Dominating Set

Definition (Realizable profile)

(T, D, M) is realizable with $S \subseteq V(G)$ if:

1. $S \subseteq \cup_{x \in T} x(G)$
2. for all $x \in V(G_i)$, $x \in D$ iff $x(G) \cap S \neq \varnothing$
3. for all $x \in V(G_i)$, $x \in M$ iff $x(G)$ is dominated by S

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k-Dominating Set

Assume that u, v are contracted to z to form G_i from G_{i+1}

k-Dominating Set

Definition (Consistency of profiles)

A profile (T, D, M) such that $z \in T$ is consistent with a set $\{(T_1, D_1, M_1), ..., (T_\ell, D_\ell, M_\ell)\}$ if the following holds. Let $T' = T \backslash \{z\} \cup \{u, v\}, D' = \cup_{j=1}^{\ell} D_j$ and $M' = \cup_{j=1}^{\ell} M_j$.

1. $T_1, ..., T_\ell$ are the red components of T' in G_{i+1}
2. $D \backslash \{z\} = D' \backslash \{u, v\}$
3. $z \in D$ iff $u \in D'$ or $v \in D'$
4. For every $x \in T \backslash \{z\}$, $x \in M$ iff $x \in M'$ or there is $y \in D'$ such that xy is a black edge in G_{i+1}
5. $z \in M$ iff for each $x \in \{u, v\}$, either $x \in M'$ or there is $y \in D'$ such that xy is a black edge in G_{i+1}

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k-Dominating Set

Algorithm: Input: a graph $G = (V, E)$, an integer $k \geq 1$ and a contraction sequence $(G_n, \ldots, G_1)$ of G with maximum red degree d

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- Initialize a map τ_n that takes as input profiles (T, D, M) with $|T| < (d^2 + 1)k$ and outputs either None if the profile is not realizable, or outputs a set $S \subseteq_{x \in T} x(G)$ that realizes (T, D, M). Initially, $\tau_n(\{v\}, \{v\}, \{v\}) = \{v\}$, $\tau_n(\{v\}, \varnothing, \varnothing) = \varnothing$ and None on every other input.

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Algorithm: Input: a graph $G = (V, E)$, an integer $k \geq 1$ and a contraction sequence $(G_n, ..., G_1)$ of G with maximum red degree d

- Let u, v be the vertices contracted in $G_{i+1} \rightarrow G_i$ into z

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- Let (T, D, M) be a profile. There are two cases:
  - $z \notin T$: then $\tau_i(T, D, M) = \tau_{i+1}(T, D, M)$

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- Let u, v be the vertices contracted in $G_{i+1} \to G_i$ into z
- Let (T, D, M) be a profile. There are two cases:
  - $z \in T$: inspect every set $\mathcal{P}$ of k-profiles consistent with (T, D, M)

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- Let (T, D, M) be a profile. There are two cases:
  - $z \in T$: inspect every set $\mathcal{P}$ of k-profiles consistent with (T, D, M)
  - Put $\tau_i(T, D, M) =$ the best $\cup_{P \in \mathcal{P}} \tau_{i+1}(P)$ (the smallest one). If all these unions are None, $\tau_i(T, D, M) =$ None

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  - Put $\tau_i(T, D, M) =$ the best $\cup_{P \in \mathcal{P}} \tau_{i+1}(P)$ (the smallest one). If all these unions are None, $\tau_i(T, D, M) =$ None
  - If there is a $\mathcal{P}$ with a profile (T, D, M) with $|T| \geq (d^2 + 1)k$, since τ_{i+1} is undefined on this profile, we instead pick $v \in T \backslash (\cup_{x \in D} B^2_{i+1}(x))$ and make the query at $(T \backslash \{v\}, D \backslash \{v\}, M \backslash \{v\})$

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k-Dominating Set

Theorem

This algorithm is correct

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Theorem

This algorithm runs in time $O(d^{2(d^2+1)k-2}2^{2(d^2+1)k})$ *where* d *is the maximum red degree of the trigraphs in the contraction sequence*

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Conclusion

Thank you for listening!

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