

Online Algorithms

Anil Maheshwari

anil@scs.carleton.ca
School of Computer Science
Carleton University
Canada

Bipartite Matching Problem

Randomized Ranking Algorithm

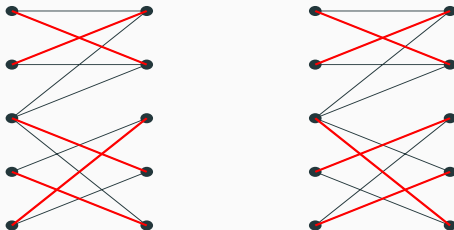
Economics-Based Analysis of RANKING Algorithm

Bipartite Matching Problem

Bipartite Matching

Let $G = (V = L \cup R, E)$ be a bipartite graph where the vertex set V consists of the sets L and R (referred to as 'left' and 'right' sets) and a set E of edges (v, w) where $v \in L$ and $w \in R$.

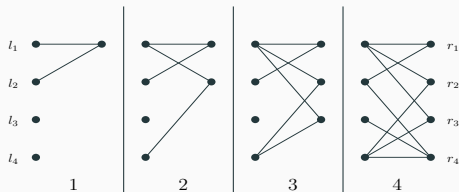
The set $M \subseteq E$ is a matching in G if no two edges in M share a vertex.



Online Matching Problem

Input: All the vertices in the set L are known in advance, but the vertices in R and the edges are presented over time. At each time instant $t \in \{1, 2, 3, \dots\}$, a new vertex $r_t \in R$ and all its incident edges arrive.

Task: The online matching algorithm needs to decide among all the currently unmatched neighbors of r_t in the set L to which vertex (if any) r_t should be matched. The vertex r_t remains matched to that vertex for the rest of the algorithm.



Output: Find a matching M of the largest possible size. Find M such that the ratio $\frac{|M|}{|M^*|}$ is as large as possible, where M^* is (offline) maximum matching in G .

Competitive Ratio

Ratio of the value returned by an Online Algorithm in comparison to the (best) Offline algorithm.

What is the largest value of $c \leq 1$, such that

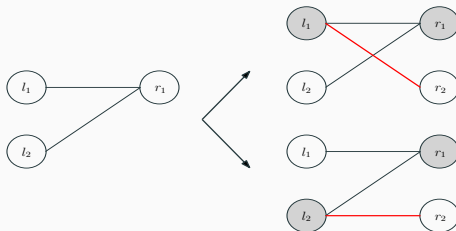
$$\frac{\text{Value (online algorithm)}}{\text{Value (off-line algorithm)}} \geq c$$

Oblivious (Non-adaptive) Adversary

Adversary must fix the graph and the arrival sequence of vertices in advance (before the algorithm's random coin flips occur). The adversary cannot adapt its choices based on the random decisions the algorithm makes during execution. Thus, the adversary cannot adaptively choose the graph.

Lower Bound on Deterministic Algorithms

Consider a bipartite graph on 4 vertices, where $L = \{l_1, l_2\}$ and $R = \{r_1, r_2\}$. At the first time step the algorithm is presented with the vertex r_1 and the two incident edges (r_1, l_1) and (r_1, l_2) .



The (non-adaptive oblivious) Adversary may have chosen the graph which forces the Competitive Ratio $= \frac{1}{2}$.

Greedy Bipartite Matching Algorithm

Greedy Online Matching Algorithm:

At time step t :

Match r_t to any of the unmatched neighbors in the set L

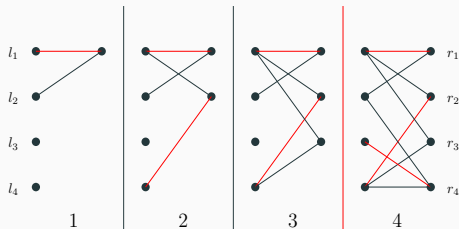


Figure 1: Competitive Ratio: $\frac{|M|}{|M^*|} = \frac{3}{4}$

Exercise: Show that the Greedy Online Matching Algorithm has a competitive ratio $\geq \frac{1}{2}$.

Randomized Ranking Algorithm

Randomized Online Bipartite Matching

Bipartite graph be $G = (V = L \cup R, E)$

Vertices in $L = \{l_1, \dots, l_n\}$ are known in advance

Vertices in $R = \{r_1, \dots, r_n\}$ arrive online along with its incident edges in increasing order of their indices

An edge is either in the matching or it isn't

Randomized RANKING Algorithm

The randomized RANKING algorithm for matching in the bipartite graph $G = (L \cup R, E)$ is as follows.

RANKING Algorithm [KVV90]

Step 1: For each vertex $v \in L$:

Assign a rank (i.e. a real number) $\text{rank}(v)$ selected independently and uniformly at random from $[0, 1]$

Step 2: For each vertex $w \in R$ in order of its appearance:

Match w to its lowest ranked unmatched neighbor (if any) in L

Economics-Based Analysis of RANKING Algorithm

Items: Vertices in $L = \{l_1, \dots, l_n\}$.

Price: Assign a price p_i to each vertex l_i by choosing a number $w \in [0, 1]$ uniformly and independently at random and set $p_i = e^{w-1}$.

Buyers: Vertices in $R = \{r_1, \dots, r_n\}$.

They arrive online with edges to the set L suggesting which items they are interested in buying.

Value: r_j gives a value of 1 to each vertex in its neighborhood. I.e.,

$$v_j(l_i) = \begin{cases} 1, & \text{if } l_i \in N(r_j) \\ 0, & \text{if } l_i \notin N(r_j) \end{cases}$$

Utility: Buyer r_j derives from buying item $l_i \in N(r_j)$:

$$u_j(l_i) = v_j(l_i) - p_i.$$

Social Welfare: Matching M corresponds to edges (l_i, r_j) if buyer r_j chooses item l_i . Let $l_i = M(j)$ indicates that l_i is matched to r_j in M . Social welfare of M is the sum total of all Buyer's valuations. I.e., $SW(M) = \sum_{j=1}^n v_j(M(j))$.

Buyers Utility and Sellers Revenue

To find, expected value of $SW(M)$, social welfare is decomposed into sum of buyers utilities and sellers revenues.

Sellers Revenue: Consider seller that sells item l_i . It earns a revenue of

$$rev_i = \begin{cases} p_i, & \text{if the item was purchased} \\ 0, & \text{otherwise} \end{cases}$$

Buyers Utility: For buyer r_j , its utility is given by

$$util_j = \begin{cases} 1 - p_i, & \text{if } r_j \text{ purchases } l_i \in N(r_j) \\ 0, & \text{if } r_j \text{ doesn't purchase any items} \end{cases}$$

We observe that for each edge $e = (l_i, r_j)$ in matching M :

$$util_j + rev_i = 1 - p_i + p_i = 1.$$

Social welfare as $\sum$ Buyers utilities + Sellers revenues

Observation 1

Consider the matching M obtained with the given choice of prices $(p_1, \dots, p_n)$. The social welfare obtained by summing the buyers utilities and the total revenues is given by

$$SW(M) = \sum_{r_j \in R} util_j + \sum_{l_i \in L} rev_i = \sum_{(l_i, r_j) \in M} (1 - p_i) + p_i = |M|$$

Lemma

For every edge $e = (l_i, r_j)$ in the bipartite graph G
 $E[\text{util}_j + \text{rev}_i] \geq 1 - \frac{1}{e}.$

Proof Sketch:

- Fix price vector $(p_1, \dots, p_n).$
- Let (l_i, r_j) be an edge in $G.$
- Let M be the matching constructed by the algorithm.
- Execute the algorithm for the same price vector with item l_i removed.
- Let M_{-i} be the matching obtained with l_i removed.
- Let $p = e^{y-1}$ be the price of the item to which r_j is matched in $M_{-i}.$ In case r_j isn't matched, set $p = 1.$

Observation 2

If $p_i < p$, then Item l_i is matched (sold) in M .

Proof Sketch: There are two cases.

Case 1: If r_j is matched in M_{-i} : For sure it will be matched in M . It can be matched to l_i in M as r_j gets a better utility since $p_i < p$, or l_i may be bought by some buyer who arrived before r_j . In any case, l_i gets sold.

Case 2: If r_j isn't matched in M_{-i} to anybody: Then $p = 1$ by assumption. In M , if l_i is still not being bought before the arrival of r_j , r_j will buy it as it gets a better utility since $p_i < p$. Therefore, l_i is also sold in this case.

Observation 3

$$util_j \geq 1 - p.$$

Proof Sketch:

- Note that $1 - p$ is the utility of buyer r_j in M_{-i} .
- When we introduce the new item l_i , utilities for any of the buyers can't decrease.
- Use induction, and reason by showing that for each buyer, we have the previous set of items available, and possibly more.
- This holds for 1st buyer, 2nd buyer, . . .

Proof of Main Lemma

Lemma

For every edge $e = (l_i, r_j)$ in the bipartite graph G
 $E[util_j + rev_i] \geq 1 - \frac{1}{e}.$

Proof Sketch: Let $y \in [0, 1]$ such that $p = e^{y-1}$. Let $I(\cdot)$ denotes indicator r.v.

$$\begin{aligned} E[rev_i] &= E[p_i \cdot I(l_i \text{ is sold})] \\ &\geq E[p_i \cdot I(p_i < p)] \quad (\text{Observation 2}) \\ &= \int_0^y e^{x-1} dx \\ &= e^{y-1} - \frac{1}{e} \\ &= p - \frac{1}{e} \end{aligned}$$

From Observation 3, we get

$$E[util_j + rev_i] \geq 1 - p + p - \frac{1}{e} = 1 - \frac{1}{e} \quad \square$$

Theorem

$E[SW(M)] \geq (1 - \frac{1}{e}) |M^*|$, where M^* is an optimal matching in G . I.e., Ranking algorithm is $(1 - \frac{1}{e})$ -competitive.

Proof Sketch: Let M^* be an optimal matching in the graph G .

- Observation 1 states that the social welfare obtained by summing the buyers utilities and the total revenues is given by

$$\sum_{r_j \in R} util_j + \sum_{l_i \in L} rev_i = \sum_{(l_i, r_j) \in M} (1 - p_i) + p_i = |M|$$

- Also,

$$\sum_{r_j \in R} util_j + \sum_{l_i \in L} rev_i \geq \sum_{(l_i, r_j) \in M^*} (util_j + rev_i)$$

as for M^* we are potentially summing the subset of quantities from $util_1 + \dots + util_{|R|} + rev_1 + \dots + rev_{|L|}$. Note that if M^* is a perfect matching then $\geq$ is $=$.

Main Result(contd.)

Now we have:

$$\begin{aligned} E[|M|] &= E \left[\sum_{r_j \in R} util_j + \sum_{l_i \in L} rev_i \right] \\ &\geq E \left[\sum_{(l_i, r_j) \in M^*} (util_j + rev_i) \right] \\ &= \sum_{(l_i, r_j) \in M^*} E[util_j + rev_i] \\ &\geq \left(1 - \frac{1}{e}\right) |M^*| \quad \square \end{aligned}$$

1. Richard Karp, Umesh Vazirani, and Vijay Vazirani, An optimal algorithm for on-line bipartite matching. ACM-STOC, 1990.
2. Alon Eden, Michal Feldman, Amos Fiat, Kinneret Segal: An Economics-Based Analysis of RANKING for Online Bipartite Matching. SOSA 2021: 107-110.