Background Quiz[1]

Instructions. Choose the *best* answer for each question. Unless stated otherwise, assume logarithms are base 2. Answers/hints towards the end..

1. Suppose an algorithm satisfies

$$T(n) = 2T(n/2) + n \log n, \qquad T(1) = \Theta(1).$$

What is the asymptotic running time of the algorithm?

(A) $\Theta(n \log n)$
(B) $\Theta(n \log^2 n)$
(C) $\Theta(n^{3/2})$
(D) $\Theta(n^2)$

2. You are given a sorted array of n distinct integers. Which of the following tasks can be solved in worst-case $\Theta(\log n)$ time?

(A) Finding the median
(B) Determining whether a given key occurs in the array
(C) Finding the maximum element
(D) Computing the sum of all elements

3. Suppose $G = (V, E)$ is a connected undirected graph with $|V| = n$ and $|E| = n - 1$. Which statement must be true?

(A) G contains exactly one cycle
(B) G is a tree
(C) Every vertex has degree at least 2
(D) G has a Hamiltonian path

4. A graph has 12 vertices, each of degree 5. How many edges does it have?

(A) 30
(B) 36
(C) 60
(D) Such a graph cannot exist

5. How many binary strings of length 8 contain exactly three 1's?

(A) 24
(B) 56
(C) 64
(D) 336

[1] AI Generated. You can generate yourself MCQ questions in algorithms, graphs, probability, and combinatorics for practice.

6. Let A and B be events with

$$\Pr(A) = 0.6, \qquad \Pr(B) = 0.5, \qquad \Pr(A \cap B) = 0.3.$$

Which statement is correct?

(A) A and B are independent
(B) A and B are mutually exclusive
(C) $\Pr(A \mid B) = 0.3$
(D) $\Pr(A \cup B) = 0.7$

7. Let $X_1, \ldots, X_n$ be indicator random variables, where $X_i = 1$ if the ith student in a class passes an exam and $X_i = 0$ otherwise. Suppose $\Pr(X_i = 1) = 0.8$ for every i, but the events need not be independent. If

$$X = \sum_{i=1}^{n} X_i,$$

what is $\mathbb{E}[X]$?

(A) 0.8
(B) $0.8n$
(C) n
(D) It cannot be determined without independence

8. To prove the statement

"If n^2 is even, then n is even,"

which of the following is a valid proof strategy?

(A) Assume n is even and prove n^2 is even
(B) Prove the contrapositive: if n is odd, then n^2 is odd
(C) Give several examples of even values of n
(D) Assume both n and n^2 are even

9. Which statement best describes proof by contradiction?

(A) Assume the conclusion and derive the hypothesis
(B) Assume the statement is false and derive an impossibility
(C) Verify the statement for sufficiently many examples
(D) Prove a stronger statement only for the base case

10. Which statement about comparison-based sorting is correct?

(A) Every comparison-based sorting algorithm requires $\Omega(n^2)$ comparisons
(B) There exists a comparison-based sorting algorithm using $O(n)$ comparisons in the worst case
(C) Every comparison-based sorting algorithm requires $\Omega(n \log n)$ comparisons in the worst case
(D) Merge sort requires $\Theta(n^2)$ comparisons in the worst case

11. Suppose a simple undirected graph has 7 vertices. What is the maximum possible number of edges?

(A) 7
(B) 14
(C) 21
(D) 49

12. A committee of 3 people is chosen uniformly at random from 5 women and 4 men. What is the probability that the committee contains exactly 2 women?

(A) $\dfrac{\binom{5}{2}\binom{4}{1}}{\binom{9}{3}}$

(B) $\dfrac{\binom{5}{2}}{\binom{9}{3}}$

(C) $\dfrac{5^2 \cdot 4}{9^3}$

(D) $\dfrac{\binom{5}{2}\binom{4}{1}}{9^3}$

13. Suppose a randomized algorithm succeeds with probability at least $1/2$ on each independent run. It is run independently 4 times, and we accept if at least one run succeeds. What is the minimum probability that the overall procedure succeeds?

(A) $\dfrac{1}{2}$

(B) $\dfrac{3}{4}$

(C) $\dfrac{15}{16}$

(D) 1

14. Which of the following statements about NP-completeness is correct?

(A) Every problem in NP is known to have a polynomial-time algorithm

(B) If one NP-complete problem has a polynomial-time algorithm, then every problem in NP has a polynomial-time algorithm

(C) Every NP-hard problem belongs to NP

(D) NP-complete problems cannot be solved by exponential-time algorithms

15. Suppose you want to prove by induction that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

for all integers $n \geq 1$. In the inductive step, after assuming the formula for $n = k$, which expression should be simplified to prove the case $n = k + 1$?

(A) $\dfrac{k(k+1)}{2} + k$

(B) $\dfrac{k(k+1)}{2} + (k+1)$

(C) $\dfrac{(k+1)(k+2)}{2} + 1$

(D) $k(k+1) + (k+1)$

16. A medical condition occurs in 1% of a population. A test is positive with probability 0.95 when a person has the condition and with probability 0.05 when the person does not have it. If a randomly selected person tests positive, what is the probability that the person has the condition?

(A) $\frac{1}{20}$

(B) $\frac{19}{118}$

(C) $\frac{19}{20}$

(D) $\frac{95}{99}$

17. Two cards are selected uniformly at random, without replacement, from a standard 52-card deck. Given that at least one of the two cards is an ace, what is the probability that both cards are aces?

(A) $\frac{1}{51}$

(B) $\frac{1}{33}$

(C) $\frac{2}{13}$

(D) $\frac{1}{17}$

18. Two fair six-sided dice are rolled. Given that their sum is at least 10, what is the probability that at least one die shows 6?

(A) $\frac{1}{2}$

(B) $\frac{2}{3}$

(C) $\frac{5}{6}$

(D) $\frac{11}{12}$

19. A permutation of $\{1, 2, \ldots, n\}$ is chosen uniformly at random. Let X be the number of indices i for which the permutation places i in position i. What is $\mathbb{E}[X]$?

(A) $\frac{1}{n}$

(B) 1

(C) $\log n$

(D) $n/2$

20. Each of n people independently chooses one of m days uniformly at random. Let X be the number of unordered pairs of people who choose the same day. What is $\mathbb{E}[X]$?

(A) $\frac{n}{m}$

(B) $\frac{n(n-1)}{m}$

(C) $\frac{n(n-1)}{2m}$

(D) $m\left(1-\left(1-\frac{1}{m}\right)^n\right)$

21. In the random graph $G(n,p)$, every possible edge is included independently with probability p. Let X be the number of isolated vertices. What is $\mathbb{E}[X]$?

(A) $n(1-p)$
(B) np^{n-1}
(C) $n(1-p)^{n-1}$
(D) $\binom{n}{2}(1-p)$

22. Breadth-first search is run from a source vertex s in an undirected, unweighted graph represented by adjacency lists. Which statement is correct?

(A) It finds minimum-weight paths even when edges have arbitrary positive weights
(B) It finds paths with the minimum number of edges in $\Theta(|V|+|E|)$ time
(C) It always produces a minimum spanning tree in $\Theta(|E|\log|V|)$ time
(D) Its worst-case running time is $\Theta(|V||E|)$

23. Dijkstra's algorithm is implemented using adjacency lists and a binary min-heap on a graph with n vertices and m edges, all of whose edge weights are nonnegative. What is its worst-case running time?

(A) $\Theta(n+m)$
(B) $O(n^2)$ for every graph
(C) $O((n+m)\log n)$
(D) $O(m\log m\log n)$

24. What is the worst-case running time for constructing a binary heap from an arbitrary array of n keys using the standard bottom-up Build-Heap algorithm?

(A) $\Theta(\log n)$
(B) $\Theta(n)$
(C) $\Theta(n\log n)$
(D) $\Theta(n^2)$

25. A binary min-heap is stored in the array

$$[2,5,4,9,7,8].$$

First, key 3 is inserted. Then Extract-Min is performed. Which array represents the resulting heap?

(A) $[3,5,4,9,7,8]$
(B) $[3,4,5,9,7,8]$
(C) $[4,5,3,9,7,8]$
(D) $[5,7,4,9,8,3]$

Answer Key and Brief Explanations

1. **(B).** By the Master Theorem, $a = 2$, $b = 2$, so $n^{\log_b a} = n$. Since $f(n) = n \log n$, the recurrence solves to $\Theta(n \log^2 n)$.
2. **(B).** Binary search takes $\Theta(\log n)$ time. The median and maximum are directly accessible in $\Theta(1)$ time in a sorted array, while summing all elements requires $\Theta(n)$ time.
3. **(B).** A connected graph with n vertices and exactly $n - 1$ edges is a tree.
4. **(A).** By the Handshaking Lemma,

$$2|E| = \sum_{v \in V} \deg(v) = 12 \cdot 5 = 60,$$

so $|E| = 30$.
5. **(B).** Choose the three positions occupied by 1's:

$$\binom{8}{3} = 56.$$

6. **(A).** Since

$$\Pr(A)\Pr(B) = 0.6 \cdot 0.5 = 0.3 = \Pr(A \cap B),$$

the events are independent. Also $\Pr(A \cup B) = 0.8$.
7. **(B).** Linearity of expectation does not require independence:

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = 0.8n.$$

8. **(B).** The contrapositive is logically equivalent to the original implication.
9. **(B).** In contradiction, one assumes the statement to be proved is false and derives an impossibility.
10. **(C).** The comparison decision-tree lower bound gives $\Omega(n \log n)$ comparisons in the worst case.
11. **(C).** A simple graph has at most one edge per unordered pair of vertices:

$$\binom{7}{2} = 21.$$

12. **(A).** Choose 2 of the 5 women and 1 of the 4 men, then divide by the number of 3-person committees:

$$\frac{\binom{5}{2}\binom{4}{1}}{\binom{9}{3}}.$$

13. **(C).** The probability all four runs fail is at most

$$\left(\frac{1}{2}\right)^4 = \frac{1}{16}.$$

Thus success probability is at least $1 - \frac{1}{16} = \frac{15}{16}$.
14. **(B).** If any NP-complete problem is solvable in polynomial time, then $P = NP$, and therefore every problem in NP is polynomial-time solvable.

15. (B). Using the induction hypothesis,

$$1 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1),$$

which simplifies to $\frac{(k+1)(k+2)}{2}$.

16. (B). By Bayes' rule, the required probability is

$$\frac{(0.01)(0.95)}{(0.01)(0.95) + (0.99)(0.05)} = \frac{0.0095}{0.059} = \frac{19}{118}.$$

17. (B). There are $\binom{52}{2} - \binom{48}{2} = 198$ two-card hands containing at least one ace, and $\binom{4}{2} = 6$ containing two aces. Hence the conditional probability is $6/198 = 1/33$.

18. (C). The ordered outcomes with sum at least 10 are

$$(4,6), (5,5), (6,4), (5,6), (6,5), (6,6).$$

Five of these six outcomes contain a 6, so the probability is 5/6.

19. (B). Let X_i indicate that i is in position i. Since $\Pr(X_i = 1) = 1/n$, linearity of expectation gives

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = n\left(\frac{1}{n}\right) = 1.$$

20. (C). For each of the $\binom{n}{2}$ unordered pairs, let an indicator record whether the pair chose the same day. Each indicator has expectation $1/m$. Thus

$$\mathbb{E}[X] = \binom{n}{2}\frac{1}{m} = \frac{n(n-1)}{2m}.$$

21. (C). A fixed vertex is isolated precisely when all of its $n-1$ possible incident edges are absent, which occurs with probability $(1-p)^{n-1}$. Therefore, by linearity of expectation,

$$\mathbb{E}[X] = n(1-p)^{n-1}.$$

22. (B). Breadth-first search explores vertices by increasing number of edges from s. With adjacency lists, every vertex and edge is processed only a constant number of times, giving running time $\Theta(|V| + |E|)$.

23. (C). There are at most n heap deletions and at most m decrease-key operations, each costing $O(\log n)$ in a binary heap. Thus the total time is $O((n+m)\log n)$.

24. (B). Although a single sift-down may take $\Theta(\log n)$ time, most nodes are near the leaves. Summing the work over all nodes gives $\Theta(n)$ total time.

25. (A). After inserting 3, the heap is $[2, 5, 3, 9, 7, 8, 4]$. Extracting 2 moves 4 to the root, and one sift-down swap with 3 yields $[3, 5, 4, 9, 7, 8]$.