Simulating the CVM Algorithm for Counting Distinct Elements in a Data Stream

By: Nader El-Ghotmi

Introduction

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The trivial solution: Store every distinct element in a data structure such as a hashmap.

Motivation: This solution scales proportionally to the number of distinct elements. Can we develop an algorithm with space which does not scale with the number of distinct elements?

The CVM Algorithm

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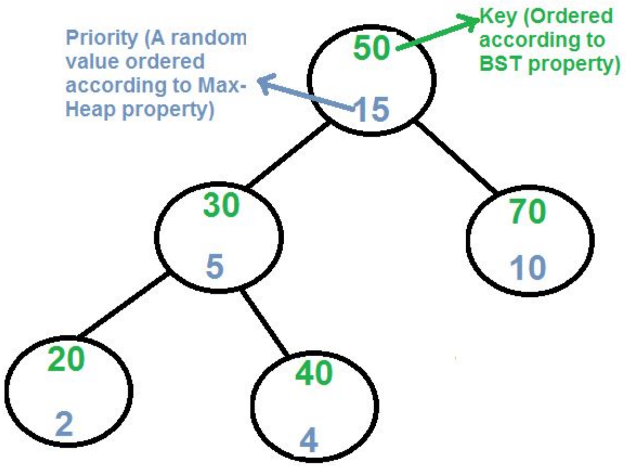
Provides an unbiased estimation for the number of distinct elements in a data stream.

Before we get started

Binary Search Tree + Max Heap

Keys sorted like BST, Values sorted like Max Heap

Get Max Value: O(1)
Search Key: O(logn)
Insert: O(logn)
Delete: O(logn)



Pseudocode for the CVM Algorithm

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Time Complexity of the CVM Algorithm

Clearly we do n iterations for each element in the stream.

So O(nT(F(s))) where F(s) is what goes on inside each iteration.

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$$T(F(s)) = \begin{cases} 2log(s) & \text{if } |B| \leq s \text{ and } a \in |B| \text{ and } u < p \\ log(s) & \text{if } |B| < s \text{ and } a \notin |B| \text{ and } u < p \\ log(s) & \text{if } |B| = s \text{ and } a \in |B| \text{ and } u \geq p \\ 2log(s) & \text{if } |B| = s \text{ and } a \notin |B| \text{ and } u < p \text{ and } u' \leq u \end{cases}$$

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$$O(nlog(s))$$

Correctness of the CVM Algorithm

To prove it is a valid estimator we need to prove that the following property is held.

$$Pr(a_j \in B_t) = p_t,\ \forall j \in\ [1, t]$$

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Unfortunately, we do not know the competitive ratio/upper bound on error.

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Experimentation

Without a competitive ratio, experimentation can still provide us with an idea of what to expect.

Experimentation - Step 1

Generating data streams for testing

Stream 1: This data stream was designed by generating N random 7 digit integers.

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Stream 6: This stream is composed of the same integer repeated save for 0.1% of the stream which are unique values.

Experimentation - Step 2

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$$[1, 3, 7, 15, 28, 53, 95, 168, 291, 500]$$

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In the case of the analysis I used 1000 data points in between 1 and 1 million.

Experimentation - Step 3

Running the CVM Algorithm for every buffer size in the list on Stream 1.

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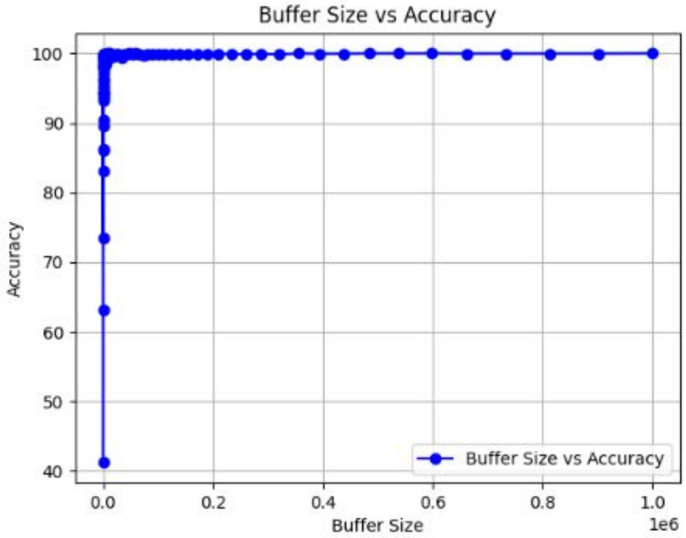

Figure 1: Buffer Size vs Accuracy for Stream 1 of size 1 million

Experimentation - Step 4

Truncating the charts to 30 data points.

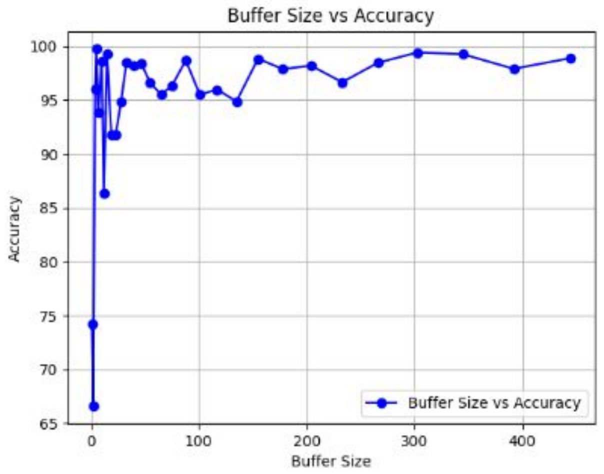

Figure 2: Buffer Size vs Accuracy for Stream 4 of size 1 million

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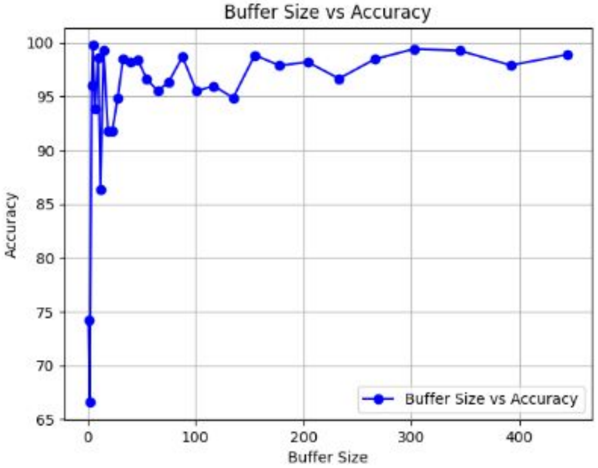

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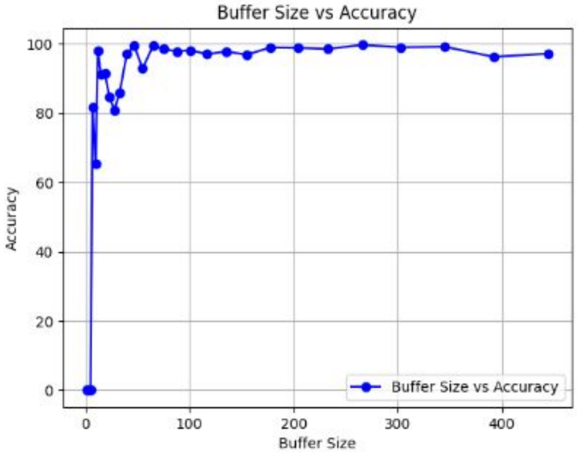

Figure 3: Buffer Size vs Accuracy for Stream 5 of size 1 million

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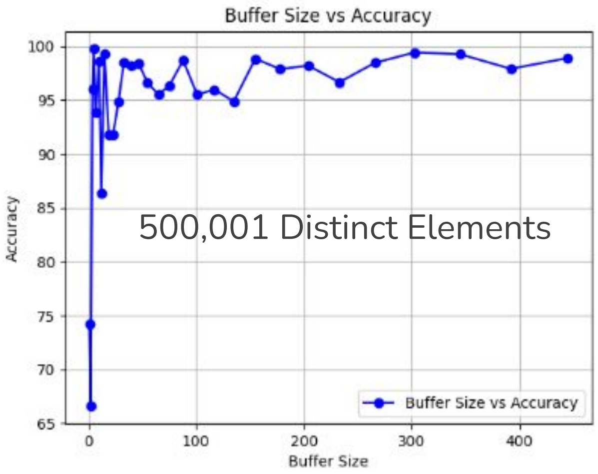

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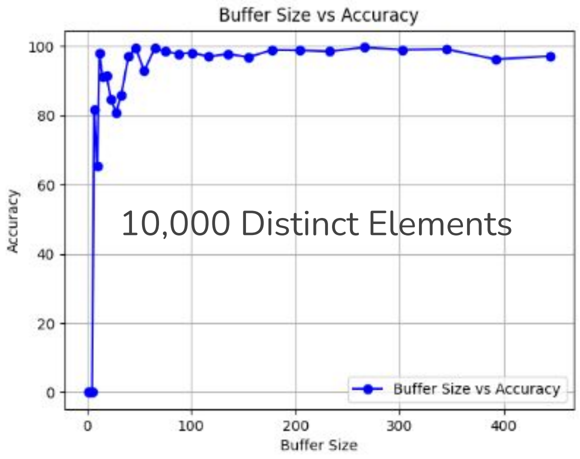

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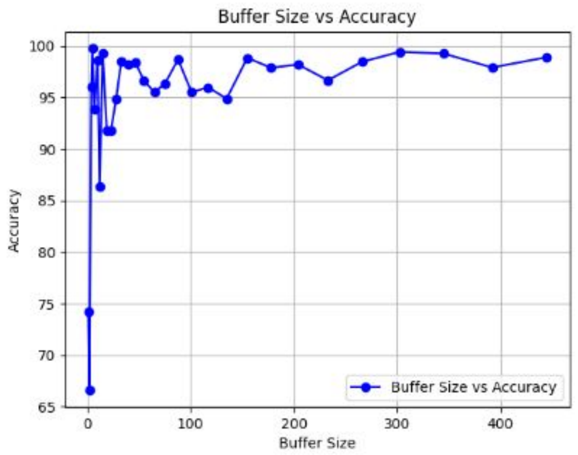

Figure 4: Buffer Size vs Accuracy for Stream 4 of size 1 million

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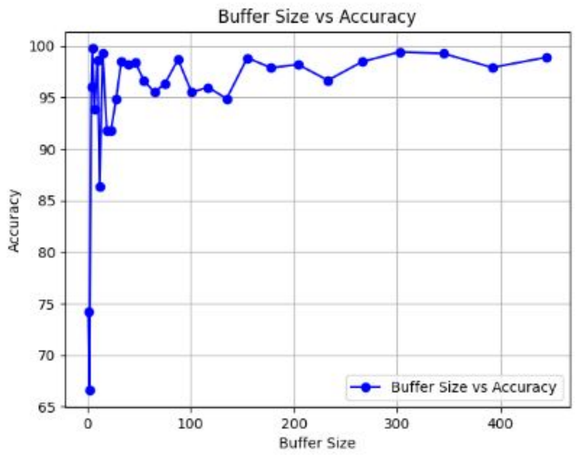

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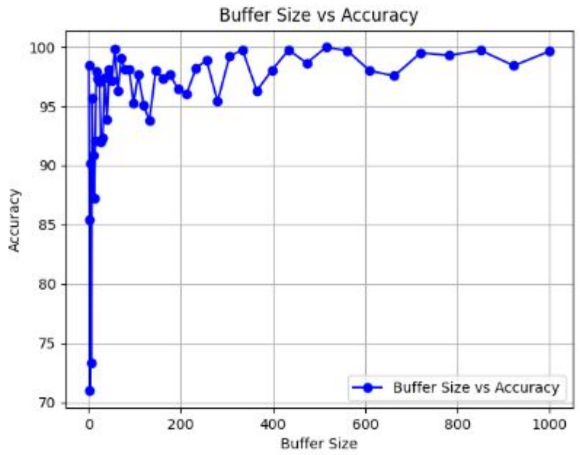

Figure 5: Buffer Size vs Accuracy for Stream 4 of size 10 million

Experimentation - Step 6

Checking the variance.

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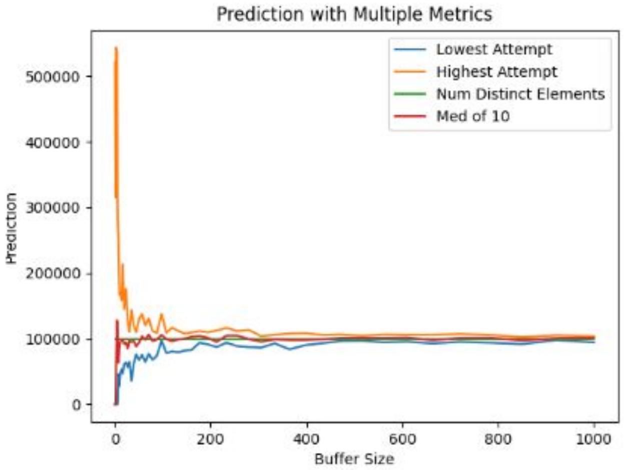

Figure 6: Buffer Size vs Best and Worst Case for Stream 5 of size 10 million

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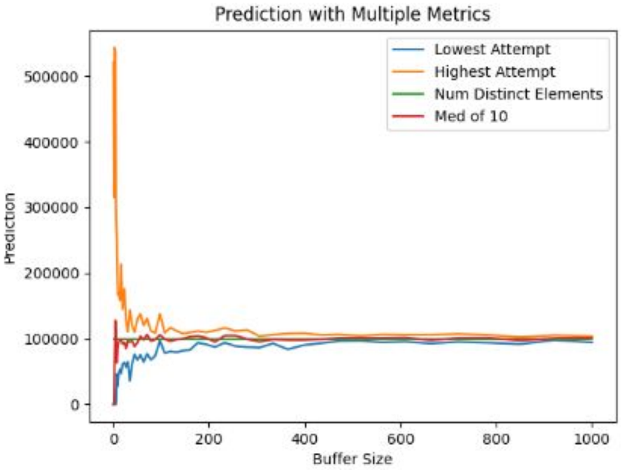

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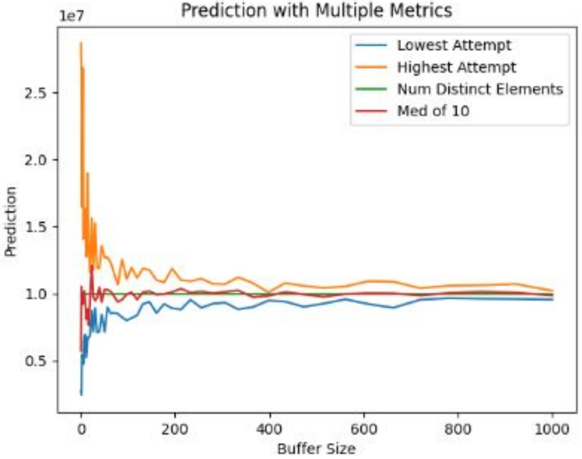

Figure 7: Buffer Size vs Best and Worst Case for Stream 3 of size 10 million

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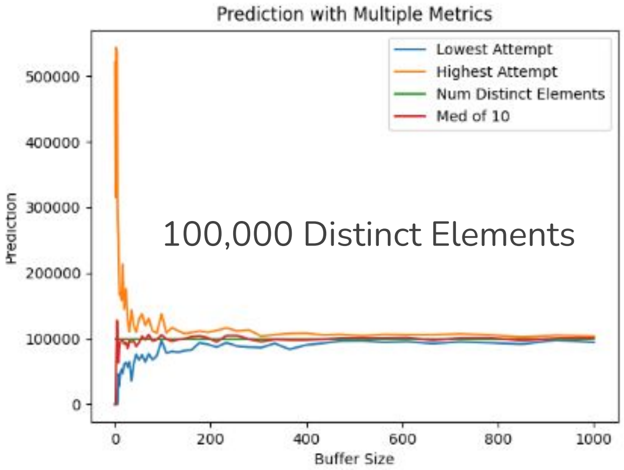

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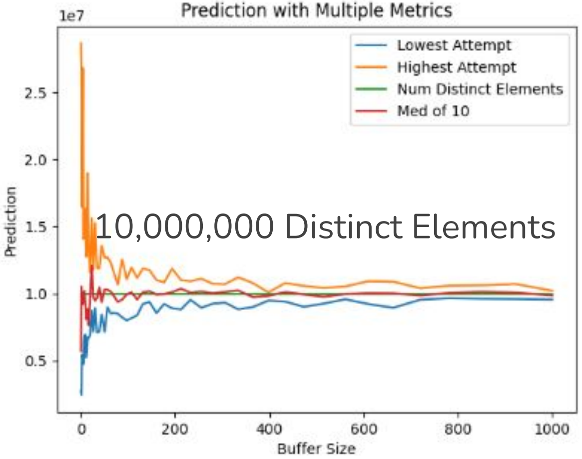

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Conclusion

Summary: The CVM Algorithm provides a simple approach to estimating the number of distinct elements in a data stream. It lacks a proven competitive ratio, but useful information can be observed through experimentation.

Plans for the future:

- Implement in C++
- Develop a more rigorous plan for analysis
- Generate streams that mimic real world examples
- Try to define the competitive ratio

References

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Thank you!

Any questions?