Traveling Salesman Problem

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Overview

Motivation

Problem Statement

Methods

- ILP Formulations
- Held-Karp
- Ant Colony Optimization
- 2-opt
- Nearest Addition
- Christofides

Comparison

Motivation

- One of the most famous problems in Discrete Optimization
- Origins Unclear
- Related to the Hamilton Cycle Problem
- NP-complete

Asymmetric TSP

Given a set of n cities, and weights between every pair Cij

Need to visit each *once* which is called a *tour*

Objective: Find the tour with minimal sum of weights between each consecutive pair of cities

Metric TSP

Subset of Asymmetric TSP, where vertices are on metric space:

- Symmetric: Cij = Cji for all i,j
- Satisfies Triangle Inequality: Cik+Ckj <= Cij

Euclidean TSP

Subset of the Metric TSP where:

- Each city is on a Euclidean plane
- Cij is the distance using the Euclidean norm

ILP Formulation: Dantzig, Fulkerson and Johnson

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to

$$\forall i : \sum_{j=1}^{n} x_{ij} = 1 \qquad \text{Every vertex has one incoming edge}$$

$$\forall j : \sum_{i=1}^{n} x_{ij} = 1 \qquad \text{Every vertex has one outgoing edge}$$

$$\sum_{i \in S} \sum_{j \in S} x_{ij} \leq |S| - 1 \qquad S \subset V, S \neq \{\}$$

$$x_{ij} \in \{0, 1\}$$

ILP Formulation: Miller, Tucker and Zemlin

$$\min \sum_{j}^{n} \sum_{i \neq j}^{n} x_{ij} w_{ij}$$

Minimize the sum of weights of edges travelled

subject to

$$\forall i : \sum_{i} x_{ij} = 1$$

$$\forall j : \sum_{i} x_{ij} = 1$$

$$\forall i > 1, j > 1 : u_i - u_j + n x_{ij} \leq n - 1$$

Travel to each vertex only once

$$u_1 = 1$$

Arbitrarily pick first vertex

$$\forall i > 1 : 1 \leq u_i \leq n$$

Held-Karp

Published in 1962 by Michael Held and Richard M. Karp (also independently Richard Ernest Bellman)

Exact method for the Asymmetric Problem

O(n^2 2^n) time complexity

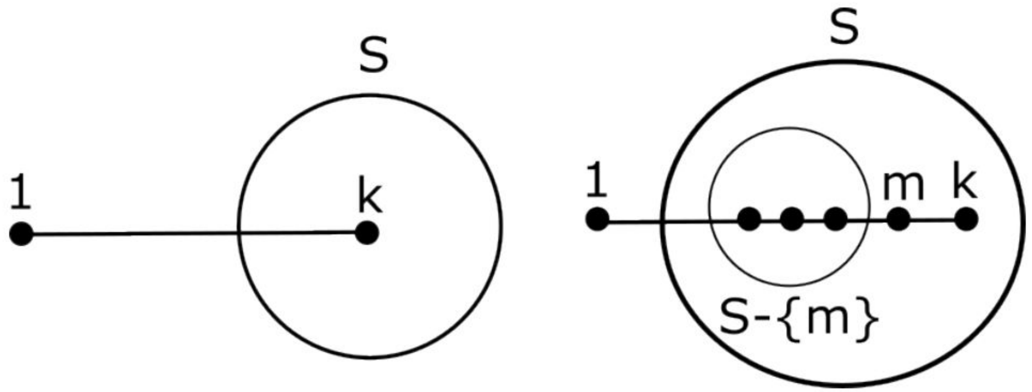
O(n 2^n) space complexity

Held-Karp

Pick first city arbitrarily

Build up shortest paths from the first city to the last (n-1 choices)

Start with smallest subsets of the problem and build up



Held-Karp

for $k \in \{2, ..., n\}$ **do**
    $g(\{k\}, k) \leftarrow c_{1k}$
end for
for $s \in \{2, ..., n-1\}$ **do**
    for $S \subset \{2, ..., n\}$ where $|S| = s$ **do**
        for $k \in S$ **do**
            $g(S, k) \leftarrow \min_{v_m \neq v_k, v_m \in S}[g(S - \{k\}, v_m) + c_{mk}]$
        end for
    end for
end for
return $\min[g(\{2, ..., n\}, k) + c_{k1}]$

Ant Colony Optimization

Devised by Marco Dorigo in 1990s first to solve the TSP (Ant System) and then extended to other problems

Convergence proved for value only (no time guarantee) and only for fixed decrease in pheremones each cycle

Solves the metric version of the TSP

Ant Colony Optimization

A number of ‘ants’ all start on random cities

Each builds up solution incrementally

Chooses edge based on probability (combination of distance and pheromone)

At end edges are reinforced with pheromone (proportional to tour length)

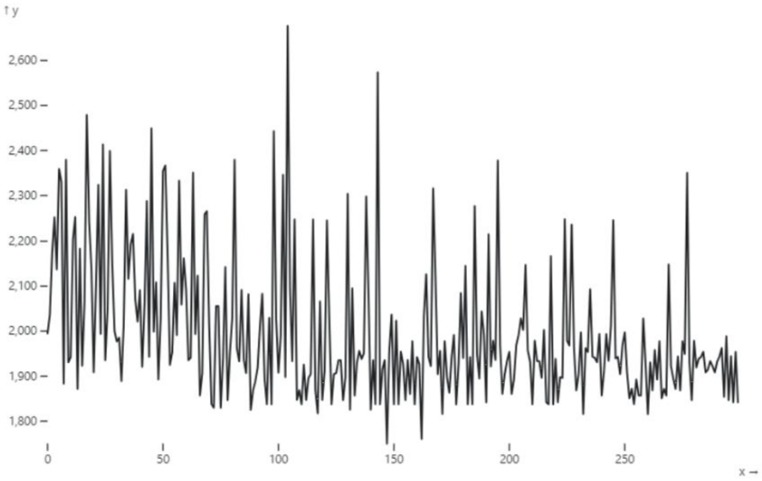
Pheromone dissipates each cycle

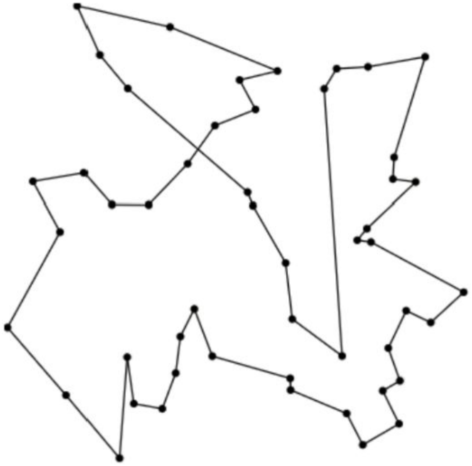
$$p_{ij}^{k} = \frac{[\tau_{ij}]^{\alpha}[\nu_{ij}]^{\beta}}{\sum_{l \in \mathcal{N}_i^k} [\tau_{ij}]^{\alpha}[\nu_{ij}]^{\beta}}$$

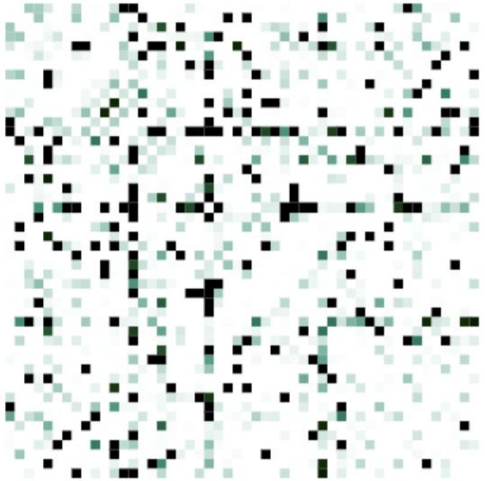
Ant Colony Optimization

```
Set parameters, initialize pheromone trails
while ¬ Termination condition do
    ConstructAntSolutions
    ApplyLocalSearch
    UpdatePheromones
end while
```

Ant Colony Optimization







2-Opt

Local Search method due to G. A. Croes, 1958

Part of a family of k-opt methods

For metric problem: ¼ $N^{(1/(2k))}$-optimal

For Euclidean problem: worst case O(logn) due to Barun, Howard and Craig

2-Opt

Start with an arbitrary tour

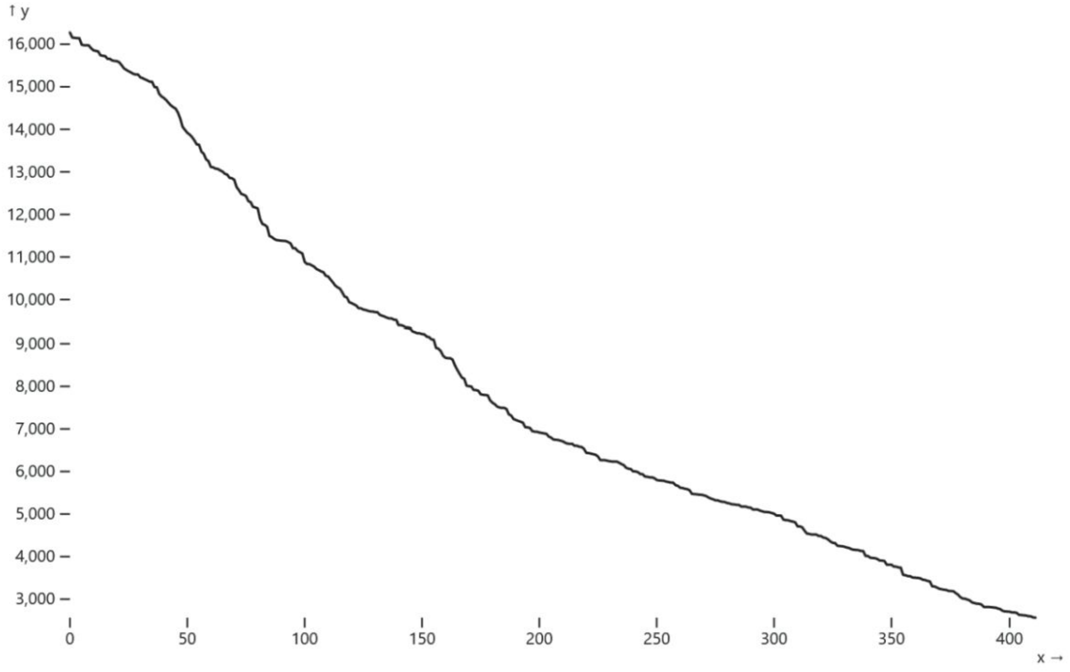
Try reversing the order of an interval of the tour for every possible interval

Replace our tour with the new tour each time we find a better one, and start over

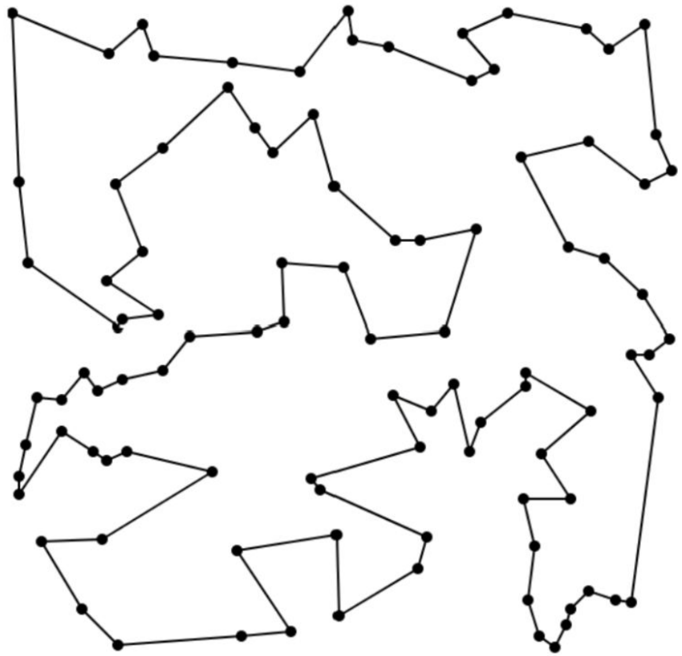
2-Opt

$T \leftarrow 1, 2, ..., n$
Start:
for $i \in \{1, 2, ..., n\}$ **do**
    for $j \in \{1, 2, ..., n\} - \{i\}$ **do**
        $T' \leftarrow$ 2-OPT(i, j)
        if $d(T') < d(T)$ **then**
            $T \leftarrow T'$
            goto Start
        end if
    end for
end for

2-Opt



↑ y
16,000
15,000
14,000
13,000
12,000
11,000
10,000
9,000
8,000
7,000
6,000
5,000
4,000
3,000
0
50
100
150
200
250
300
350
400
x →



Nearest Addition

Pick nearest pair of vertices and form cycle between them

Add next vertex closest to our cycle after the vertex it is closest to in the cycle

Repeat until there are no more vertices left

Nearest Addition

$i, j \leftarrow \underset{i,j \in S}{\text{argmin}}\ c_{ij}$

$S \leftarrow (i, j)$

while $|S| < |V|$ **do**

  $i, j \leftarrow \underset{i \notin S, j \in S}{\text{argmin}}\ c_{ij}$

  insert j into S after i

end while

return Cycle formed by path S returning to first vertex

Nearest Addition: 2-optimal proof

Same vertices chosen as MST

MST must be shorter than tour, since tour minus edge *is* a tree!

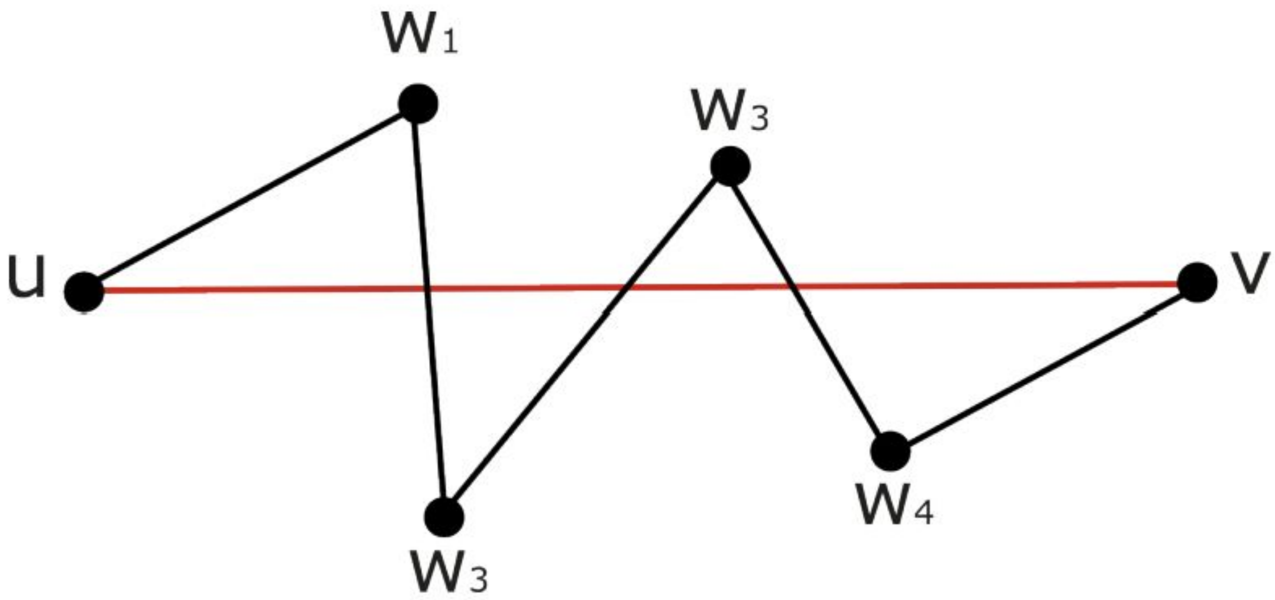
Double every edge of MST

Find Euclidean cycle for this doubled tree

Use shortcutting on cycle

Solution must be no more than twice as long as optimal

Shortcutting



Christofides

Nicos Christofides extended this idea to get a better bound in 1976

3/2 Optimal with worst case $O(n^3)$

Christofides

Instead of doubling tree, we find a perfect matching of the odd vertices

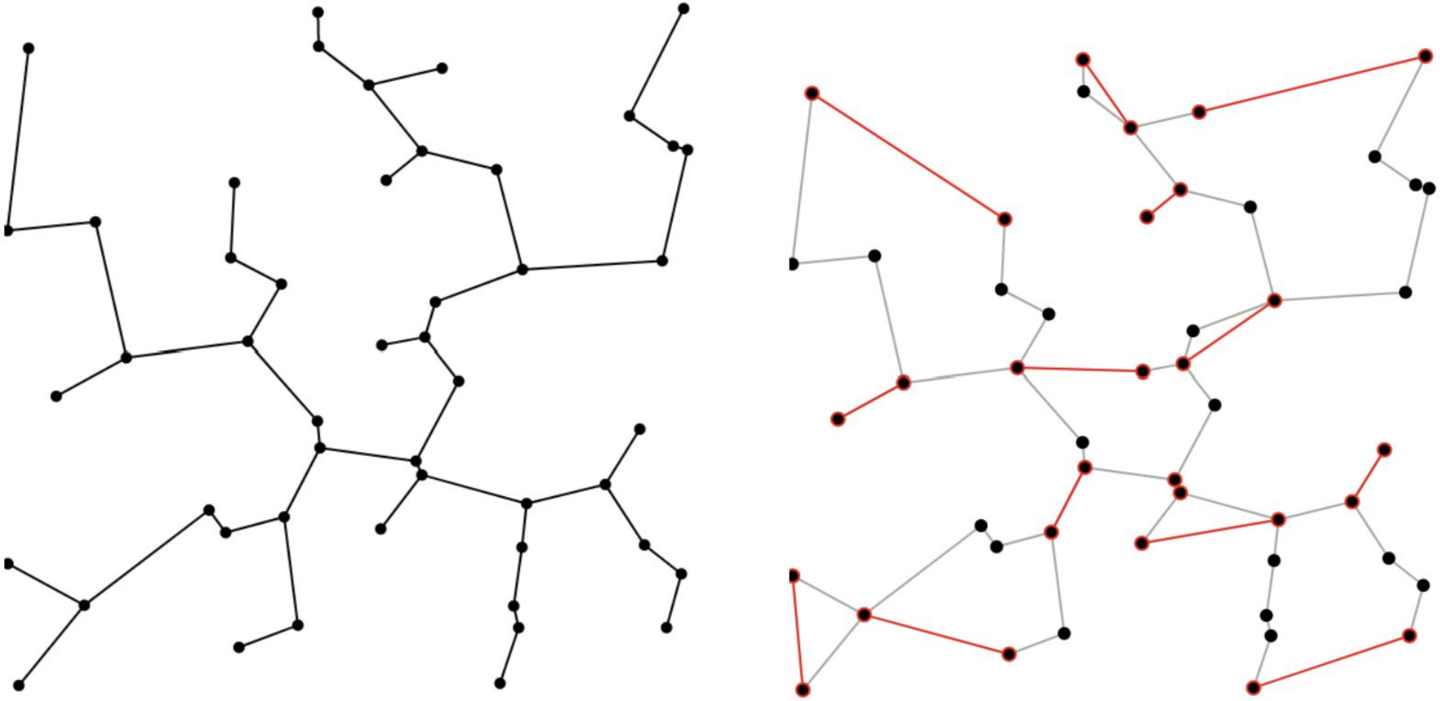
Matching can only be half as long as optimal tour

This means algorithm is 3/2 optimal!

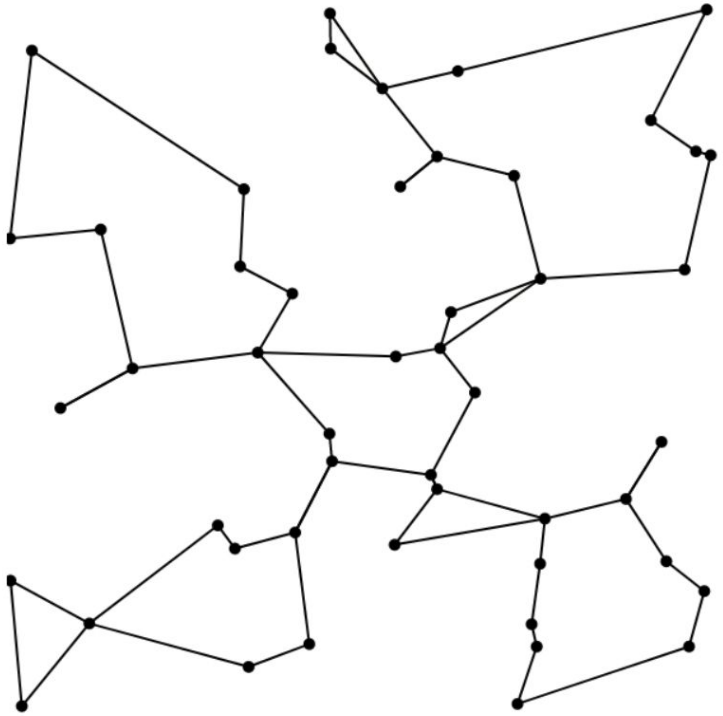
Christofides

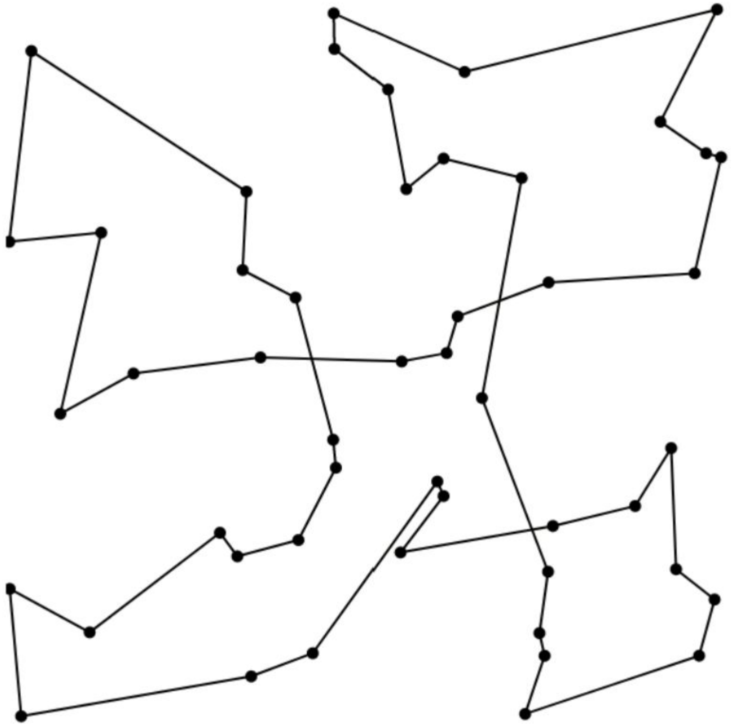
1. Find Minimum Spanning Tree T
2. Find odd-degree vertices O of T
3. Find Minimum Weight Perfect Matching M of O
4. Find Eulerian Cycle E of $M \cup T$
5. Perform shortcutting on E

Christofides

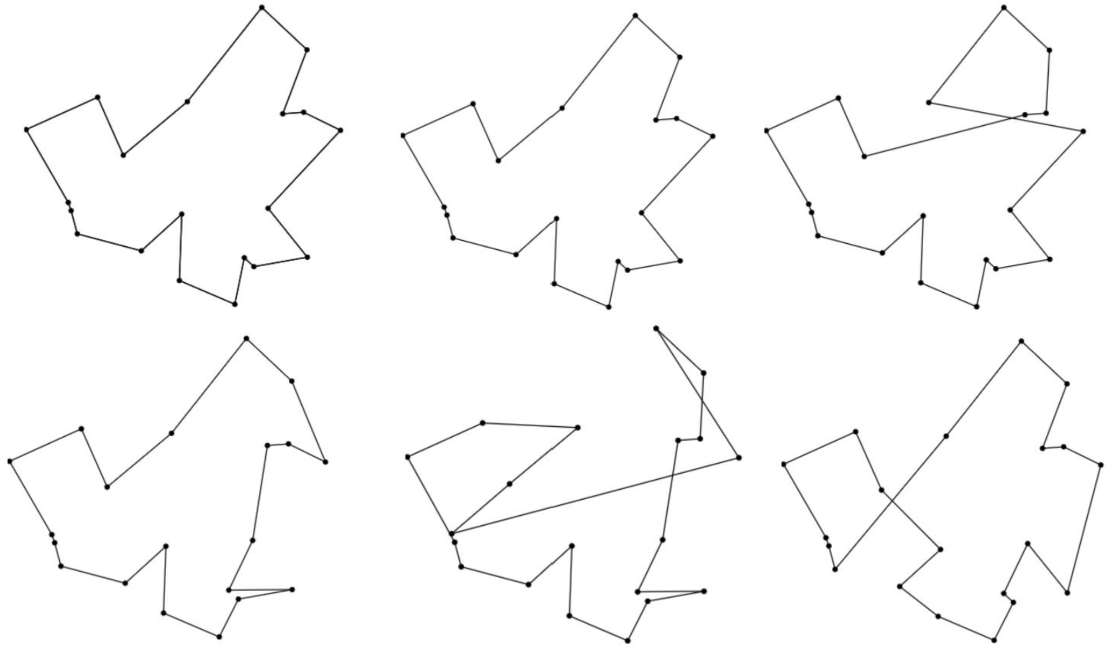


Christofides





Comparison: 20 cities



Left to right: ILP, Held-Karp, 2-opt, Christofides, Nearest Addition, ACO (α = 1.5, β = 2, ρ = 0.05, ants = 10).

Comparison: 20 cities

True value was 1111.11 (ILP and Held-Karp)

Christofides: 1153.09 (3.8%)

Nearest Addition: 1472.08 (32.5%)

2-opt: 1289.20 (16.0%)

ACO: 1211.61 (9%)

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