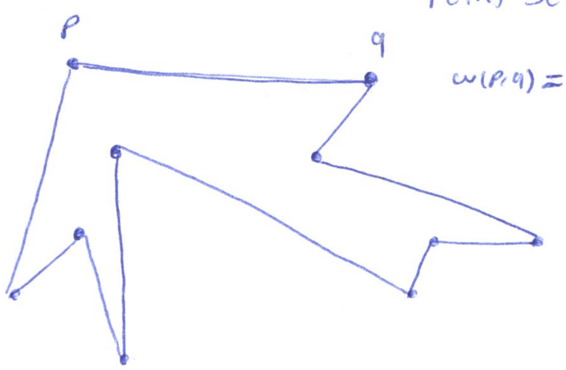
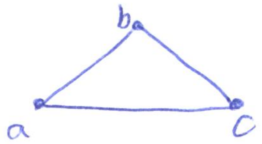
Euclidean TSP



Point set P

$w(p,q) = |pq|$

Property 1 (Triangular Inequality):



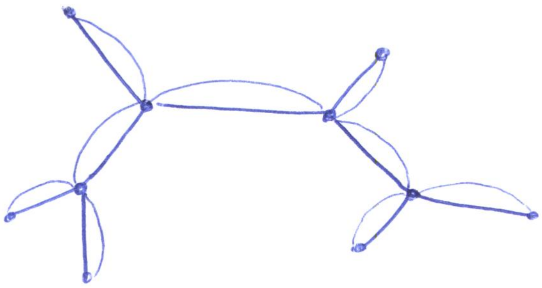
$|ac| \leq |ab| + |bc|$

Property 2 (Extention of Property 1):

$|ab| \leq |e_1| + |e_2| + \cdots + |e_m|$

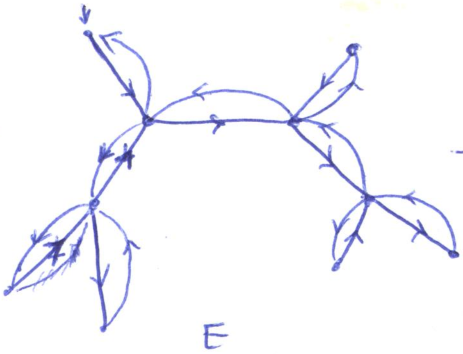
First Alg.

- Find MST(P)
- Duplicate edges of MST(P)
- Find an Eulerian tour E
- Convert E to a tour T by ~~go~~ walking through vertices of E while skipping the vertices that already visited.



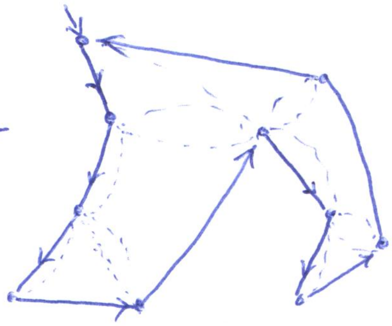
MST(P)

$\rightarrow$



E

$\rightarrow$



T

$\Downarrow$

Eulerian tour: a closed walk which visits each vertex of the graph exactly once.

Theorem: a connected graph has an Eulerian ~~toeur~~ tour iff all vertices are of even degree.

Theorem: The above alg. is a 2-approximation alg. for ETSP.

Proof:

let OPT be an optimal tour.

by removing any edge from OPT we optain a spanning path T' which is a spanning tree as well. Thus

$$\left.\begin{array}{l} T' < OPT \\ MST \leq T' \end{array}\right\} \Rightarrow MST \leq OPT \quad (1)$$ abuse the notation for weight

The weight of E is two times the weight of MST because E contains two copies of each edge in MST.

$$E = 2\,MST$$

Since we obtain T by shortcutting the edges of E, the cost of T does not exceed the cost of E (remember the triangle inequality)

$$\Rightarrow T \leq E = 2MST \quad (2)$$

Finally $\xrightarrow{(1),(2)} T \leq E = 2\,MST \leq 2 \cdot OPT$

$$\Rightarrow T \leq 2 \cdot OPT$$

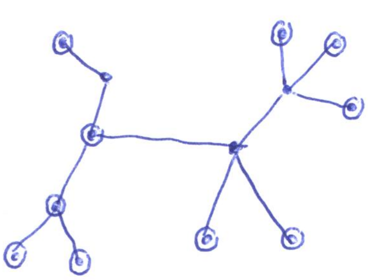
Second Alg.

- Find $MST(P)$
- Let V' be the vertices of odd degree in $MST(P)$
- Compute a minimum perfect matching M for V'
- Add edges of M to $MST(P)$
- Find an Eulerian tour E in $M \cup MST(P)$
- Convert E to a tour T

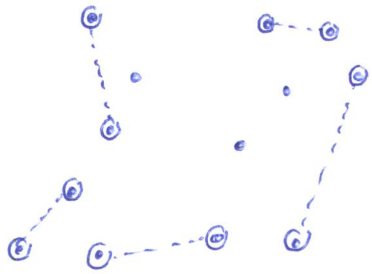
perfect matching: a matching which covers all the vertices

Observation: the number of vertices of odd degree is even.

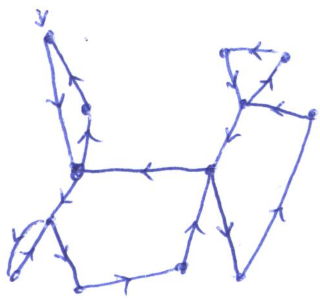
$\Rightarrow V'$ has a perfect matching



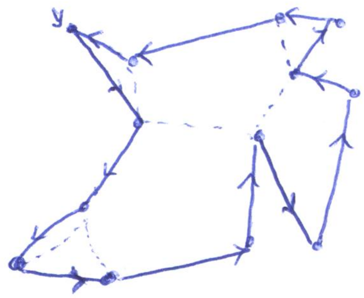
MST and V'



M



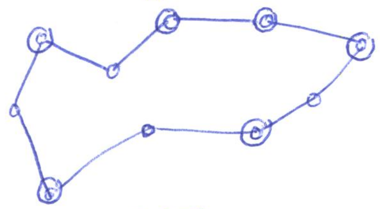
$M \cup MST$ is Eulerian



T

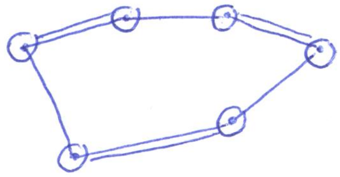
Theorem: The above alg. is a $\frac{3}{2}$-approximation for the ETSP.

Let OPT be ~~the~~ an optimal tour and let T' be the tour obtaind from OPT by shortcutting the vertices of $V \setminus V'$.



OPT

$\Rightarrow$



$T' = M_1 + M_2$

$T' \leq OPT$

T' contains two perfect matching for V' ~~say~~ M_1 and M_2

M is a minimum perfect matching and smaller than both M_1 and M_2

$M \leq M_1$ and $M \leq M_2 \Rightarrow 2M \leq M_1 + M_2 = T'$

$\Rightarrow \cancel{M \leq}$ $\qquad \Rightarrow M \leq \frac{T'}{2} \leq \frac{OPT}{2}$

$T^* \leq E = M + MST$

$E = M + MST \leq \frac{OPT}{2} + OPT = \frac{3}{2} OPT \quad (3)$

$(1),(3)$

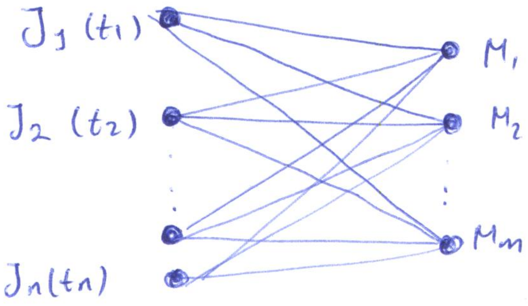
$\Rightarrow T \leq \frac{3}{2} OPT$

Load Balancing:

Given n jobs $j_1, \dots, j_n$ and m machines $M_1, \dots, M_m$.

Eeach job j_i needs time t_i to be done.

Assign jobs to machines such that ~~the~~ the time until all jobs are finished is minimized.



First Alg. (Greedy)

for $i \leftarrow 1$ to n
  assign J_i to M_k of minimum load

Theorem: The above alg. is a 2-approximation alg. for load balancing prob.

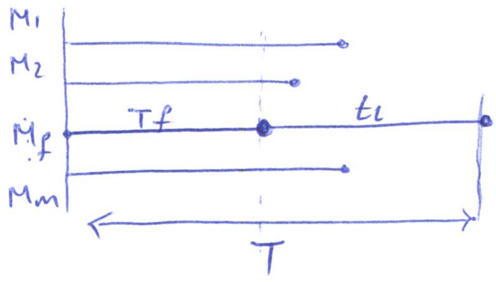
Let T be the time required by alg. and OPT be the optimal time.

$$OPT \geq t_{max} \quad \text{where } t_{max} = max(t_i)$$

$$OPT \geq \frac{1}{m} \sum_{i=1}^{n} t_i$$

At the end of alg., let M_f be the machine which maximizes the load.

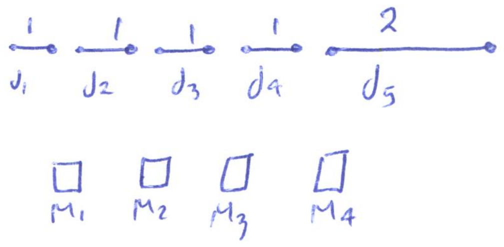
Let J_ℓ be the last job assigned to M_f.



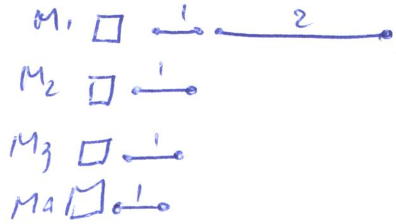
$$T = T_f + t_\ell \leq \frac{1}{m}\sum_{i=1}^{n} t_i + t_{max} \leq opt + OPT$$

$$\Rightarrow T \leq 2 \cdot OPT$$

How to improve:

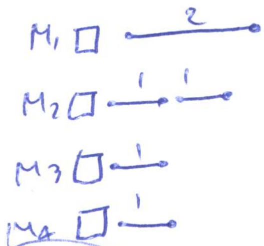


assignment 1



$T = 3$

assignment 2



$T = 2$

Observation: Its better to assign larger jobs first.

Second Alg 1.

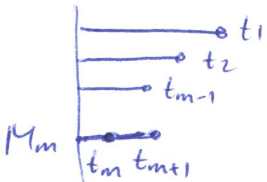
$L \leftarrow$ decreasig ordered list of jobs
for $i \leftarrow 1$ to n
  assign $L(i)$ to M_k of minimum load

Theorem: The above alg. is a $\frac{3}{2}$-approximation alg. for load balancing problem.

assume $t_1 \geq t_2 \geq \dots \geq t_n$

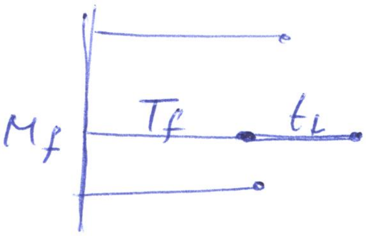
if $n \leq m$ then the alg. is optimal

if $n > m$ then $opt \geq t_m + t_{m+1}$



Let M_f be the machine with maximum load and assume J_ℓ is the last job assigned to M_f.

$$\left.\begin{array}{l} t_\ell \leq t_m \\ t_\ell \leq t_{m+1} \end{array}\right\} \Rightarrow 2t_\ell \leq t_m + t_{m+1} \Rightarrow t_\ell \leq \frac{t_m + t_{m+1}}{2}$$



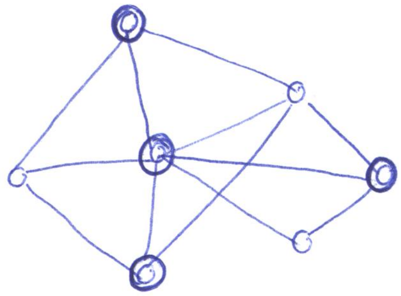
$$T = T_f + t_\ell \leq \frac{1}{m} \sum_{i=1}^{m} t_i + \frac{t_m + t_{m+1}}{2} \leq opt + \frac{OPT}{2}$$

$$\Rightarrow T \leq \frac{3}{2} \cdot OPT$$

Vertex-Cover Problem

$G=(V,E)$, a vertex-cover is a subset $C \subseteq V$ s.t. for each edge $(a,b) \in E$, either $a \in C$ or $b \in C$ (or both)

C covers all the edges of G !!!



Vertex-Cover Problem: compute a cover C^* of minimum size.

First Alg.

$C \Leftarrow \emptyset$
while $E \neq \emptyset$ do
  $(a,b) \leftarrow$ arbitrary edge in E
  $C \leftarrow C \cup \{a,b\}$
  $E \leftarrow E \setminus \{$all edges incident on u or $v\}$
return C

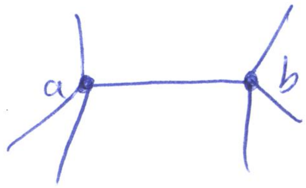
Running Time $= O(V+E)$

Theorem: The above alg. is a 2-appx alg. for Vertex-Cover problem.

Let M be the set of edges selected in the while loop.

Let C^* be an optimal vertex cover.

For each edge $(a,b) \in M$, C^* contains either a or b. $\Rightarrow |C^*| \geq |A|$



We add both of a and b to $C \Rightarrow |C| \leq 2|A|$

$\Rightarrow C \leq 2|A| \leq 2|C^*|$

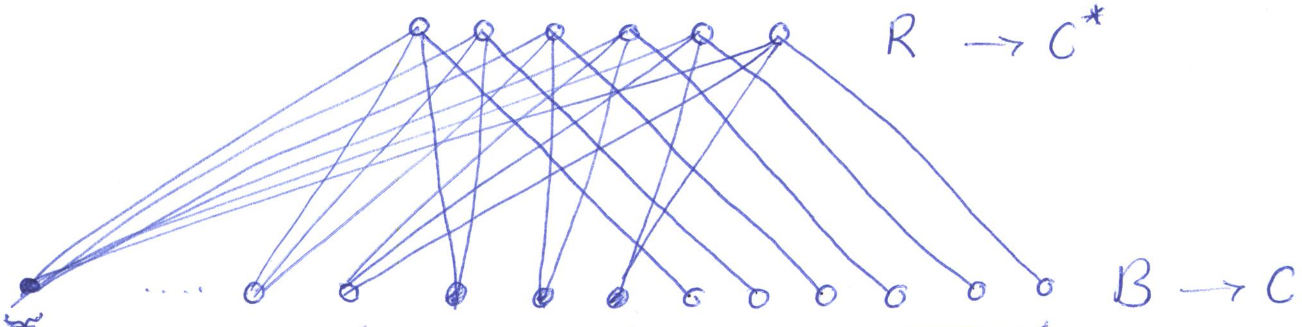
$\rightarrow$ Any maximal matching in G is a 2-appx for Vertex-Cover Problem.

Second Alg.

$C \leftarrow \emptyset$
while $E \neq \emptyset$ do
  $a \leftarrow$ vertex with maximum degree in current graph
  $C \leftarrow C \cup \{a\}$
  $E \leftarrow E \setminus \{$all edges incident on $a\}$
return C.

What is the approximation ratio α?

$\alpha = \Omega(\log n)$



deg = |R|	deg = 3	deg = 2	deg = 1
$\|R\|/\|R\| = 1$	$\|R\|/3$	$\|R\|/2$	$\|R\|$

$$|B| = \frac{|R|}{1} + \frac{|R|}{2} + \frac{|R|}{3} + \dots + \frac{|R|}{|R|} = |R| \sum_{i=1}^{|R|} \frac{1}{i} = |R| \log |R|$$

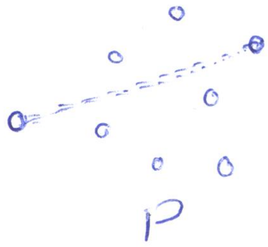
$$\left.\begin{array}{l} C = B \\ C^* = R \end{array}\right\} \Rightarrow \frac{|C|}{|R|} = \log |R| = \Theta(\log n)$$

$\Rightarrow \alpha \geq \log n$

k-clustering:

For a point set P, diameter of P, $d(P)$, is defined as the maximum distance between any pair of points in P.

$$d(P) = \max\{|pq| : p, q \in P\}$$

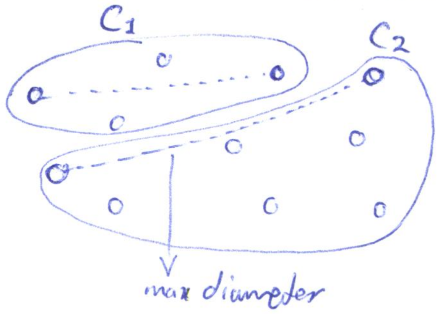


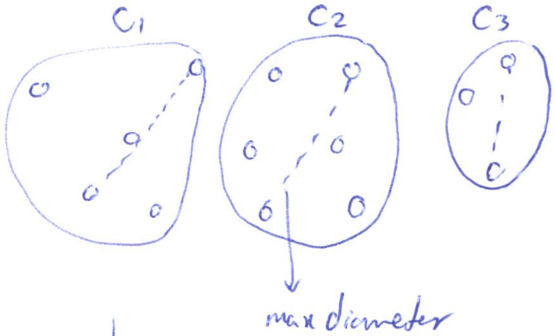
Given a point set P, we want to partition P into k clusters s.t. the diameter of clusters is minimized.

=> we want to minimize the maximum diameter.

example:

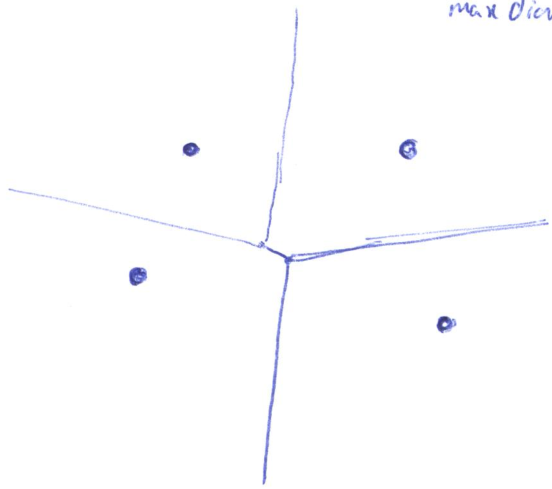
$k = 3$





Voronoi Diagram - VD

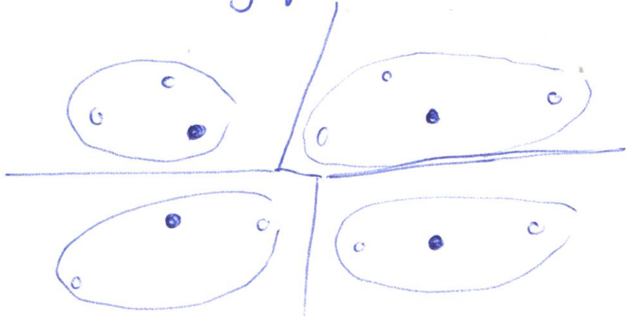
given a point set S, $VD(S)$ is defined as depicted in the picture.



Algorithm

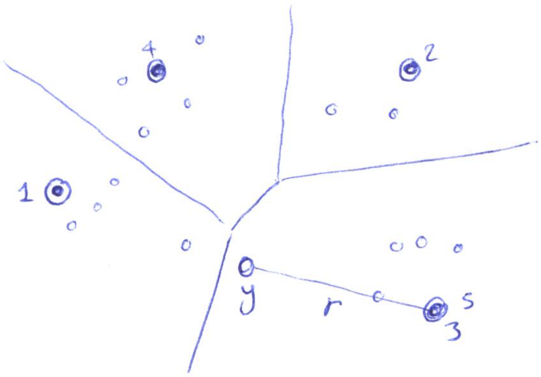
Idea: - pick k points of P as S

- compute $VD(S)$
- assign the remaining points to each voronoi cell



Algorithm

$S \leftarrow \{\text{arbitrary point of } P\}$
for $i \leftarrow 2$ to k
  $x \leftarrow$ point of P which is farthest from S
  $S \leftarrow S \cup \{x\}$
Assign the points in $P \setminus S$ to their closest point in S.



Theorem: The above alg. is a 2-appx alg. for the k-clustring problem

Let y be the next element that we have chosen by repeating the for loop ($(k+1)^{th}$ element) and let r be the distence of y to its closest point $\hat{s}$ in S.
All points in cluster s are at distance at most r of s, then the diameter of s is at most $2r$.

We have $k+1$ points ~~such~~ all of them are at distance at least r from each other (in each iteration the farthest point distance is decreasing) ant by pegion hole principle two of them fall in the same cluster in OPT. So OPT has diameter at least r.