DISJOINT PATH PROBLEM (Kleinberg+Tardos – Algorithm Design)

INPUT : A directet graph G.

k-pairs $(s_1, t_1), \ldots\ldots, (s_k, t_k)$

integer capacity $c>0$.

OUTPUT : Find the paths between as many pairs as possible so that none of the edges in G are used by more than c-pairs.

Important Special Case.

→ When $c=1$, then this is the disjoint path problem — i.e. in this case we need to find maximum number of pairs so that paths between them are edge disjoint.

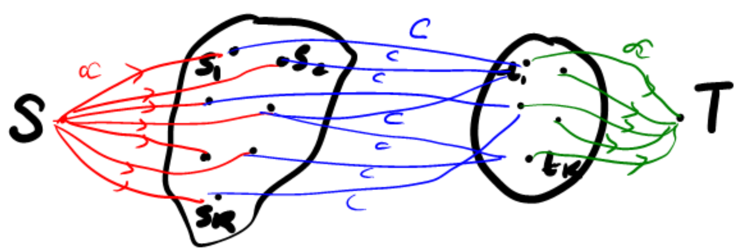
Can Max-Flow Algorithms Help?

Let set of sources $S = \{s_1, s_2, \ldots, s_k\}$

and set of sinks be $T = \{t_1, t_2, \ldots, t_k\}$

and set the capacity of each edge as c.

Solve the multiple source, multiple sink network flow problem.



Doesn't work!

[There is no way to ensure that the flow went from s_i to t_i or to some other terminal t_j.]

Disjoint Path Problem is NP-Complete !

Lets look for Approximation Algorithm

First we design a greedy algo for the disjoint case (i.e. $c=1$) and generalize it to $c>1$ case.

Following algorithm outputs a set $I \subseteq \{1,2,\dots,k\}$ and edge disjoint paths.

Greedy Disjoint Paths

Set $I := \emptyset$
$E' := E$;

While $\exists$ a path in $G' = (V, E')$ connecting a pair (s_i, t_i) which has not been connected so far **do**

1: Among all pairs (s_i, t_i) which have not been connected so far, find shortest path P_i in G' between s_i and t_i. Let i be the index corresponding to the smallest distance among them.

2: $I := I \cup \{i\}$

3: $E' := E' \setminus P_i$

Claim: Greedy-Disjoint Paths Algorithm is a $2\sqrt{|E|}+1$ approx. algorithm for max. disjoint path.

Proof: Its obvious that the paths computed by the algorithm are edge disjoint.

Consider an optimal solution.

Let I^* be the index set corresponding to optimal soln.

Let P_i^* for $i \in I^*$ be the corresponding path in optimal soln.

We will bound $|I|$ in terms of $|I^*|$.

Long paths: A path consisting of $\geq \sqrt{m}$ edges.

Short paths: A path consisting of $< \sqrt{m}$ edges.

I_s^* : Indices in I^* corresponding to short paths P_i^*

I_s : Indices in I corresponding to short paths P_i.

Claim A: There are at most $\sqrt{m}$ long paths in an optimal solution, since paths are edge disjoint.

Consider a short path in I^*.

How can $|I^*|$ be much bigger than $|I|$?

For this to happen many pairs need to be connected in I^* but not in I.

Let (s_i, t_i) be connected by a short path P_i^* in optimal, but not in greedy at all.

Why? Since in greedy (s_i, t_i) are not connected at all $\Rightarrow$ at least one of ~~the~~ edge $e \in P_i^*$ must have occured in a path P_j that was selected earlier by the greedy algo.

Note that $|P_j| \le |P_i^*| \le \sqrt{m} = \sqrt{|E|}$

Also each edge in the greedy path P_j can block at most one optimal path P_i^*.

$\Rightarrow$ Each short path P_j blocks atmost $\sqrt{m}$ paths in an optimal solution.

Hence, we get

$$|I_s^* - I| \leq \sum_{j \in I_s} |P_j| \leq |I_s| \sqrt{m}$$

Consider I^*

Long Paths $\leq \sqrt{m}$

Paths in I

short paths not in I. $\leq |I_s| \sqrt{m}$

$$\Rightarrow \quad |I^*| \leq \sqrt{m} + |I| + |I_s^* - I|$$

$$\leq \sqrt{m} + |I| + |I_s| \sqrt{m}$$

$$\leq (2\sqrt{m} + 1) |I|$$

Hence $\dfrac{|I^*|}{|I|} \leq 2\sqrt{m} + 1$ $\square$

Consider the case for $c=2$; Atmost two paths can share an edge.

Greedy-Path-Algorithm.

Set $I := \phi$; $E' := E$; $l_e := 1 \;\; \forall e \in E$; $\beta = m^{1/c+1}$.

length of each edge.

while $\exists$ a path between (s_i, t_i) which have not been connected so far in $G' = (V, E')$ do

1. Let P_i be shortest path in G' between the pair (s_i, t_i) among all the pairs that have not been connected so far.
2. $I := I \cup \{i\}$
3. Multiply the length of each edge along P_i by β.
4. Remove all edges in E' that have been used c times.

Claim: The above algorithm is a $(2cm^{1/c+1}+1)$ – approximation algorithm.

Proof: Illustrated for $c=2$.

I^* – Optimal Solution

P_i^* – Paths used in Optimal Solution

P_i^* is short if its length $\sum_{e \in P_i^*} \bar{l}(e) < \beta^2$.

I_S^* – Set of optimal short paths.

$\bar{l}(e)$ is defined to be the length of edge e when the greedy algorithm runs out of short paths.

I – Approximat Solution

P_i – Paths

P_i is short if its length $< \beta^2$

I_S = Set of short paths.

Paths Selected by Greedy Algorithms.

Claim 1: Consider a pair (s_i, t_i) such that $i \in I^*$ but is not connected by the greedy algorithm.

Then $\bar{\ell}(P_i^*) > \beta^2$.

Proof: Observe that in greedy-algo as long as short paths are selected, the condition for $c=2$ is automatically met.

(first time an edge is used its length is β, next time it is used its length is β^2, and it will not be used for any other short path after that).

Endpoints (s_i, t_i) are not connected by Greedy Algo

$\Rightarrow \nexists$ a short path joining them till the length function reaches $\bar{\ell}$.

$\Rightarrow$ Since the condition $c=2$ is not a restriction for short paths, this implies that the path P_i^* can't be a short path.

$\Rightarrow \bar{\ell}(P_i^*) > \beta^2$.

☺

Claim 2: The set I_s of short paths selected by greedy algorithm satisfy the following inequality

$$\sum_e \bar{l}_e \le \beta^3 |I_s| + m.$$

Proof: Initially the length of each edge is 1, overall its m.

A short path increases the length by at most β^3, since its length is $\le \beta^2$ and lengths of each edge along the path are increased by a factor of β.

In all there are $|I_s|$ short paths

$\boxtimes$

Claim 3: For $c=2$, greedy algo is a $(4\, m^{1/3}+1)$-approximation algorithm.

Proof: (1) $\forall\, i \in I^* - I$, $\quad \bar{\ell}(P_i^*) \geq \beta^2 \quad$ (Claim 1).

(2) $$\sum_{i \in I^* - I} \bar{\ell}(P_i^*) \geq \beta^2 \, |I^* - I| .$$

(3) Since each edge is used by at most 2-paths in I^*

$$\sum_{i \in I^* - I} \bar{\ell}(P_i^*) \leq \sum_{e \in E} 2\, \bar{l_e}$$

(4) $$\beta^2 |I^*| \leq \beta^2 |I^* - I| + \beta^2 \, |I|$$

$$\leq \sum_{i \in I^* - I} \bar{\ell}(P_i^*) + \beta^2 |I| \quad \text{(using 2)}$$

$$\overset{\text{(using (3))}}{\leq} \sum_{e \in E} 2\, \bar{l_e} + \beta^2 |I| \leq 2\,(\beta^3 |I| + m) + \beta^2 |I|$$

(5) Set $\beta = m^{1/3}$ and using the fact that $|I| \geq 1$,

$$|I^*| \leq (4\, m^{1/3} + 1)\, |I| .$$