Matrix Multiplication

$[A]_{n\times n}$; $[B]_{n\times n}$; $[C]_{n\times n}$

$C = AB$

$c_{ij} = \sum_{k=1}^{n} A[i,k]\, B[k,j]$ Requires n multiplication, $n-1$ additions

$= A[i,1] * B[1,j] + A[i,2] * B[2,j] + \cdots + A[i,n] * B[n,j]$

C Requires $\left.\begin{array}{l} n^3 \text{ multiplications} \\ n^2(n-1) \text{ additions} \end{array}\right\} O(n^3)$ time.

Can this be improved?

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

each $A_{11}, A_{12}, A_{21}, A_{22}, B_{11}, B_{12}, B_{21}, B_{22}$ are $n/2 \times n/2$ matrices.

$$C = AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{21} + A_{22}B_{21} \end{bmatrix}$$

How do we compute $A_{11}B_{11} + A_{12}B_{21}$?

Compute $A_{11}B_{11}$ and $A_{12}B_{21}$ Recursively !

"+" of two $n/2 \times n/2$ matrices can be computed in $O(n^2)$ time.

We need to perform

- 8 Recursive Calls to multiplication of two $n/2 \times n/2$ matrices.

$(A_{11}B_{11}, A_{12}B_{21}, A_{11}B_{12}, A_{12}B_{22}, A_{21}B_{11}, A_{22}B_{21}, A_{21}B_{12}, A_{22}B_{22})$

- 4 additions of two $n/2 \times n/2$ matrices

$A_{11}B_{11}+A_{12}B_{21}, \quad A_{11}B_{12}+A_{12}B_{22}, \quad A_{21}B_{11}+A_{22}B_{21}, \quad A_{21}B_{12}+A_{22}B_{22}$

Define $T(n)$ = Time to Multiply two $n \times n$ matrices

$$T(1) = 1$$

$$T(n) = 8T\left(\frac{n}{2}\right) + O(n^2)$$

$$= 8T\left(\frac{n}{2}\right) + cn^2 \quad \text{for some constant } c > 0.$$

Express $T(n)$ in terms of n.

Assume $n = 2^k$ for some positive integer k.

Then,

$$T(n) = cn^2 + 8T\left(\frac{n}{2}\right)$$

Expand

$$= cn^2 + 8\left[8T\left(\frac{n}{2^2}\right) + c\left(\frac{n}{2}\right)^2\right]$$

Simplify.

$$= cn^2\left[1 + \frac{8}{2^2}\right] + 8^2 T\left(\frac{n}{2^2}\right)$$

Expand

$$= cn^2\left[1 + \frac{8}{2^2}\right] + 8^2\left[8T\left(\frac{n}{2^3}\right) + c\left(\frac{n}{2^2}\right)^2\right]$$

Simplify

$$= cn^2\left[1 + \frac{8}{2^2} + \frac{8^2}{(2^2)^2}\right] + 8^3 T\left(\frac{n}{2^3}\right)$$

$$= cn^2\left[1 + \frac{8}{2^2} + \frac{8^2}{(2^2)^2}\right] + 8^3\left[8 \cdot T\left(\frac{n}{2^4}\right) + c\left(\frac{n}{2^3}\right)^2\right]$$ Expand

$$= Cn^2\left[1+\frac{8}{2^2}+\frac{8^2}{(2^2)^2}+\frac{8^3}{(2^3)^2}\right]+8^4T\left(\frac{n}{2^4}\right)$$ Expand

$$= cn^2\left[1+\frac{8}{2^2}+\frac{8^2}{(2^2)^2}+\frac{8^3}{(2^3)^2}\right]+8^4\left[8T\left(\frac{n}{2^5}\right)+c\left(\frac{n}{2^4}\right)^2\right]$$ Simplify

$$= Cn^2\left[1+\frac{8}{2^2}+\frac{8^2}{(2^2)^2}+\frac{8^3}{(2^3)^2}+\frac{8^4}{(2^4)^2}\right]+8^5T\left(\frac{n}{2^5}\right)$$

What will be the last term? $(n=2^k)$

$$= Cn^2\underbrace{\left[1+\frac{8}{2^2}+\frac{8^2}{(2^2)^2}+\cdots\cdots+\frac{8^{k-1}}{(2^{k-1})^2}\right]}_{\text{series sum where the next term is } \frac{8}{2\cdot 2}=2 \text{ times the previous term.}}+8^kT\left(\frac{n}{2^k}\right)$$

$$= Cn^2\left[\frac{2^k-1}{2-1}\right]+8^kT(1)$$

$$= cn^2[n-1]+(2^3)^k = cn^3-cn^2+(2^k)^3$$

$$= Cn^3-cn^2+n^3$$

$$= O(n^3)$$ $\boxtimes$ NO SAVINGS

Strassen

$$C = AB = \begin{bmatrix} M_5+M_4-M_2+M_6 & M_1+M_2 \\ M_3+M_4 & M_1+M_5-M_3-M_7 \end{bmatrix}$$

$M_1 = A_{11}(B_{12}-B_{22})$

$M_2 = (A_{11}+A_{12})B_{22}$

$M_3 = (A_{21}+A_{22})B_{11}$

$M_4 = A_{22}(B_{21}-B_{11})$

$M_5 = (A_{11}+A_{22})(B_{11}+B_{22})$

$M_6 = (A_{12}-A_{22})(B_{21}+B_{22})$

$M_7 = (A_{11}-A_{21})(B_{11}+B_{12})$

This method requires

- 7 multiplications of $n/2 \times n/2$ matrices
- 18 additions/subtractions of two $n/2 \times n/2$ matrices

$\Rightarrow$

$$T(n) = 7\,T\left(\frac{n}{2}\right) + O(n^2)$$

Express $T(n)$ in terms of n!

$$= cn^2 + 7\,T\left(\frac{n}{2}\right)$$

$$= cn^2 + 7\left[7T\left(\frac{n}{2^2}\right) + c\cdot\frac{n}{2^2}\right]$$

$$= cn^2\left[1 + \frac{7}{2^2}\right] + 7^2\,T\left(\frac{n}{2^2}\right)$$

$$= cn^2\left[1 + \frac{7}{2^2}\right] + 7^2\left[7\,T\left(\frac{n}{2^3}\right) + c\cdot\left(\frac{n}{2^2}\right)^2\right]$$

$$= cn^2\left[1 + \frac{7}{2^2} + \frac{7^2}{(2^2)^2}\right] + 7^3\,T\left(\frac{n}{2^3}\right)$$

$$= cn^2\left[1 + \frac{7}{2^2} + \frac{7^2}{(2^2)^2}\right] + 7^3\left[7T\left(\frac{n}{2^4}\right) + c\cdot\left(\frac{n}{2^3}\right)^2\right]$$

$$= cn^2\left[1 + \frac{7}{2^2} + \frac{7^2}{(2^2)^2} + \frac{7^3}{(2^3)^2}\right] + 7^4\,T\left(\frac{n}{2^4}\right)$$

The last term will be $(n = 2^k)$.

$$= cn^2\underbrace{\left[1 + \frac{7}{2^2} + \frac{7^2}{(2^2)^2} + \frac{7^3}{(2^3)^2} + \cdots + \frac{7^{k-1}}{(2^{k-1})^2}\right]}_{\text{Series sum with ratio } 7/4} + 7^k\,T\left(\frac{n}{2^k}\right)$$

$$= cn^2\left[\frac{\left(\frac{7}{4}\right)^k - 1}{\frac{7}{4} - 1}\right] + 7^k \qquad = \frac{4}{3}cn^2\left[\frac{7^k - 4^k}{4^k}\right] + 7^k$$

But $4^k = n^2$ and $7^k = 7^{\log_2 n} = n^{\log_2 7}$

$$\Rightarrow T(n) = \frac{4}{3}cn^2\left[\frac{n^{\log_2 7} - n^2}{n^2}\right] + n^{\log_2 7} = O\left(n^{\log_2 7}\right) \boxtimes$$