REPORT

A SEPARATOR THEOREM FOR PLANAR GRAPHS

Yun Luo

Section: 95.573

Abstract: The vertices of any n-vertex planar graph can be partitioned into three sets A, B, C such that no edge joins a vertex in A with a vertex in B, neither A nor B contains more than 2n/3 vertices, and C contains no more than $2\sqrt{2}\sqrt{n}$ vertices. We exhibit an algorithm which finds such a partition A, B, C in O(n) time by an algorithm.

Introduction

For many kinds of combinatorial problems, we can use divide-and-conquer" approach to divide them into some simple subproblems, and then combine the subproblem solutions to give the solution to the original problem. In order to use this approach successfully and efficiently, three things are necessary. First, the subproblems must be of the same type as the original and independent of each other. Second, the cost of solving the original problem given the solutions to the subproblems must be small. Finally, the subproblems must be significantly smaller than the original. any n-vertex graph can be partitioned into three sets A,B,C such that no edge joins a vertex in A with a vertex in B, neither A nor B contains more than α n vertices, and C contains no more than βf(n) vertices($\alpha<1$, $\beta>0$). If we can use this approach to consider those problems which are defined on graphs, For example, the sets A and B define the subproblems, we can use divide-and-conquer approach to solve a lot of those problems efficiently.

Before we talk about this separator theorem, there are some known separator theorems. For example, if we remove a single edge of any n-vertex binary tree, it can be separated into two subtrees, each with no more than 2n/3 vertices. The same situation is fitting for any n-vertex tree. A $\sqrt{n}$-separator theorem holds for the class of grid graphs, and it also holds for an one-tape Turing machine graph.

Although not all sparse graphs has an f(n)-separator theorem for some f(n)=o(n), planarity has. Before we prove it, we need to use three facts about planarity.

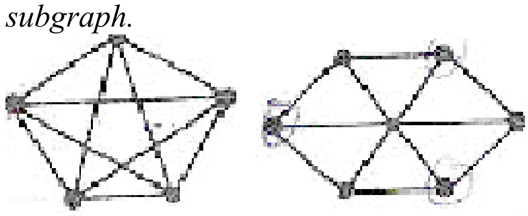
Separator theorems

In order to prove the theorems, we need to use three facts about planarity.

THEOREM 1(Jordan curve theorem). *let C be any closed curve in the plane, removal of C divides the plane into exactly two connected regions, the "inside" and the "outside" of C.*

THEOREM 2. *Any n-vertex planar graph with $n \geqslant 3$ contains no more than 3n-6 edges.*

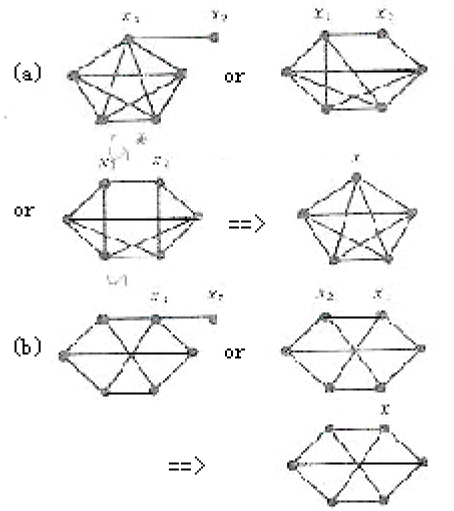
THEOREM 3(Kuratowski's theorem). *A graph is planar if and only if it contains neither a complete graph on five vertices nor a complete bipartite graph on two sets of three vertices as a generalized subgraph.*



(a) (b)

From theorem3, we can get the following lemma and its corollary.

LEMMA 1. *Let G be any planar graph, shrinking any edge of G to a single vertex preserves planarity.*



COROLLARY 1. *Let G be any planar graph. shrinking any connected subgraph of G to a single vertex preserves planarity.*

If we can consider more general situation, such as consider planar graphs which has nonnegative costs on the vertices, it will be more useful in some application. So we should prove the following lemma.

LEMMA 2. *Let G be any planar graph with nonnegative vertex costs summing to no more than one. Suppose G has a spanning tree of radius r. Then the vertices of G can be partitioned into three set A, B, C, such that no edge joins a vertex in A with a vertex in B, neither A nor B has total cost exceeding 2/3, and C contains no more than 2r+1 vertices, one is the root of the tree.*

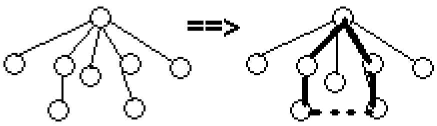
For example:

The cycle(bold line) divided the graph into two parts(inside and outside)

So A—inside of cycle

B—outside of cycle

C--cycle



LEMMA 3. *Let G be any n-vertex connected planar graph having nonnegative vertex costs summing to no more than one. Suppose that the vertices of G are partitioned into levels according to their distance from some vertex v, and that L(l) denotes the number of vertices on level l. If r is the maximum distance of any vertex from v, let r+1 be an additional level containing no vertices. Given any two levels l_1 and l_2 such that levels 0 through l_1-1 have total cost not exceeding 2/3 and levels l_2+1 through r+1 have total cost not exceeding 2/3, it is possible to find a partition A,B,C of the vertices of G such that no edge joins a vertex in A with a vertex in B, neither A nor B has total cost exceeding 2/3, and C contains no more than $L(l_1)+L(l_2)$ $+max\{0,2(l_2-l_1-1)\}$ vertices*

Proof:

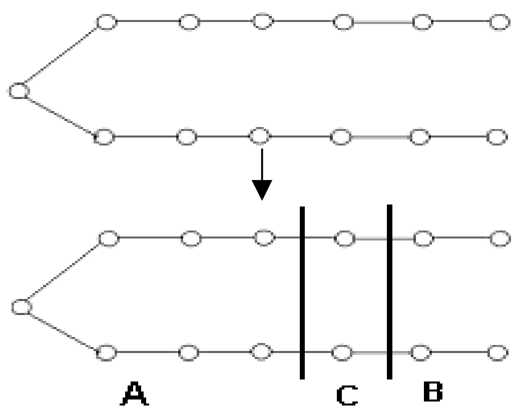
1>if $l_1 \geq l_2$

A— L(0~ l_1-1)

B— L(l_1+1~r)

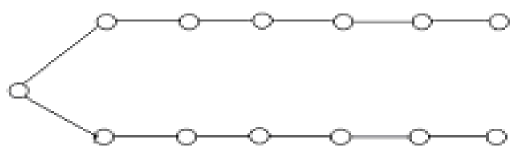
C— L(l_1)

0 l_0 l_2 $l_2+1/(l_1-1)$ l_1 l_1+1 r

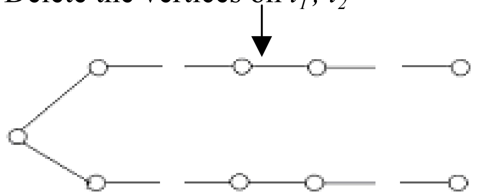


2>if $l_1 < l_2$

0 l_1-1 l_1 l_1+1 l_2-1 l_2 l_2+1



Delete the vertices on l_1, l_2



the graph will be divided into three parts, the only part which can have cost>2/3 is the middle part($l_1+1 \sim l_2-1$)

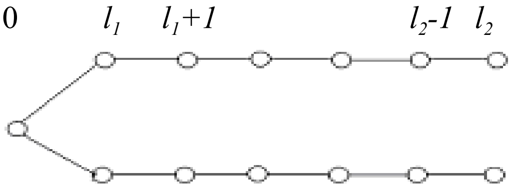
***a*>if the middle part ≤ 2/3**

A--the most costly part of the three

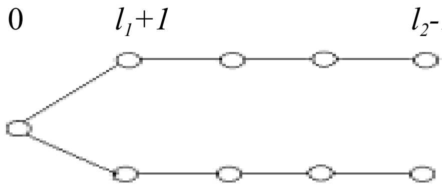
B--the remaining two parts

C--L(l_1)+ L(l_2)

b> if the middle part >2/3



delete all vertices on levels l_2 and above,shrink all vertices on levels l_1 and below to a single cost zero



apply lemma 2 to this new graph. A', B', C' are the resulting vertex partition

A—MAX{A', B'}

B---L(l_1)+ L(l_2)+(2(l_2-l_1-1)+1)-1

C—the remaining vertices in G

THEOREM 4.*Let G be any n-vertex planar graph having nonnegative vertex costs summing to no more than one. Then the vertices of G can be partitioned into three sets A,B,C such that no edge joins a vertex in A with a vertex in B, neither A nor B has total cost exceeding 2/3, and C contains no more than* $2\sqrt{2}\sqrt{n}$ *vertices*

Let us prove it.

If(connected(G))

{

Partition the vertices into levels according to their distance from some vertex v.

Let L(l)---- be the number of vertices on level l.

r--------maximum distance of any vertex from v.

levels –1 and r+1 contains no vertices.

l_1-------level(the sum costs<1/2 from level 0 through l_1–1 and the sum cost ⩾1/2 from level 0 through l_1)

k-------the number of vertices on level *0* through l_1.

Find a level l_0 ,such that $l_0 \leqslant l_1$ and $|L(l_0)| + 2(l_1 - l_0) \leqslant 2\sqrt{k}$.

Find a level l_2, such that $l_1 + 1 \leqslant l_2$ and $|L(l_2)| + 2(l_2 - l_1 - 1) \leqslant 2\sqrt{n-k}$.
If(exist(l_0) and exist(l_2))
{
By Lemma 3, the vertices of G can be partitioned into three sets A,B,C such that no edge joins a vertex in A with a vertex in B, neither A nor B has cost exceeding 2/3, and C contains no more than $2(\sqrt{k} + \sqrt{n-k}) \leqslant 2\sqrt{2}\sqrt{n}$ vertices.
}
If(!exist(l_0))
{
for $i \leqslant l_1$, $L(i) \geqslant 2\sqrt{k} - 2(l_1 - i)$ since $L(0)=1$ ==>$1 \geqslant 2\sqrt{k} - 2l_1$, and $l_1 + 1/2 \geqslant \sqrt{k}$ ==> $l_1 \geqslant \sqrt{k}$
$k = L(0)+..+L(l_1) \geqslant \sqrt{k}(\sqrt{k}+1) > k$..
}
this is a contradiction.
If(!exist(l_2))
{
similar contradiction.
}
} //end of if(connected(G))
for example: the FIG. 3 can be divided into three sets A,B,C. see FIG.4.

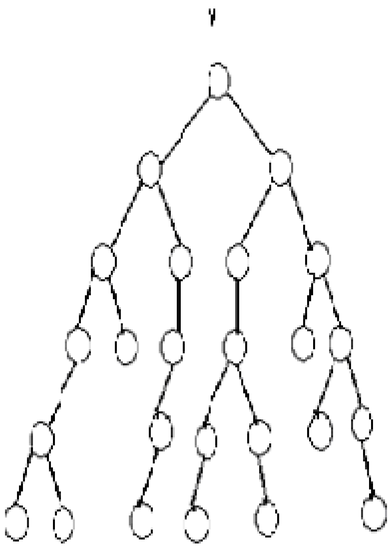


FIG.3

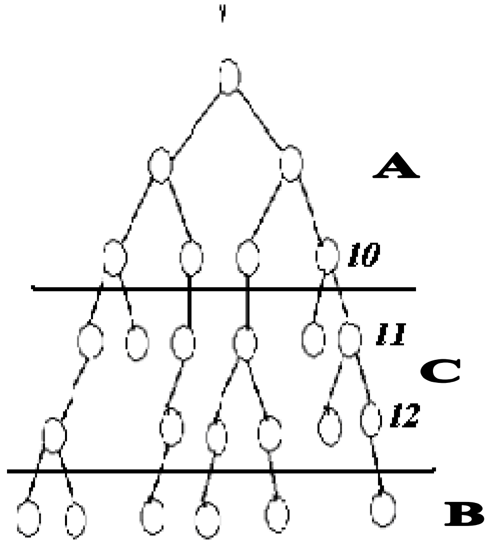


FIG.4

if(!connected(G))
{
Let G_1, G_2,..., G_k be the connected components of G, with vertex V_1, V_2,..., V_k, respectively.
If(no connected component has total vertex cost exceeding 1/3)
{
let i-------- the minimum index such that the total cost of $V_1 \cup V_2 \cup V_3 \cup \cdots \cup V_i$ exceeds 1/3
let A---- $V_1 \cup V_2 \cup \cdots \cup V_i$
let B---- $V_{i+1} \cup V_{i+2} \cup \cdots \cup V_k$
let C-----NIL
since i is minimum and the cost of $V_i \leqslant 1/3$
So the cost of A $\leqslant 2/3$
So the theorem is true.
}
for example: the FIG.5 can be divided into three sets A,B,C. see FIG.6.

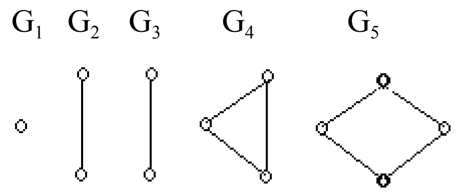


FIG.5

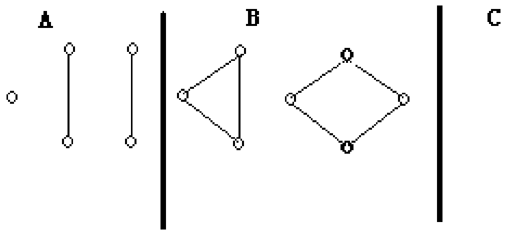


FIG.6

If(some connected component(G_i)has total vertex cost between 1/3 and 2/3)

{

let A-------V_i

let B------ $V_1 \cup V_2 \cup V_3 \cup \cdots \cup V_{i-1} \cup V_{i+1} \cup V_{i+2} \cup V_{i+3} \cup \cdots \cup V_k$

let C-----NIL

so the theorem is true

}

for example: the FIG.7 can be divided into three sets A,B,C. see FIG.8

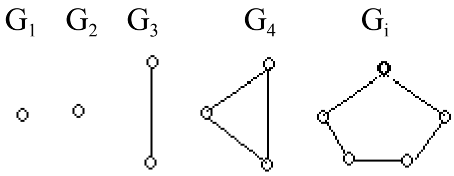


FIG.7

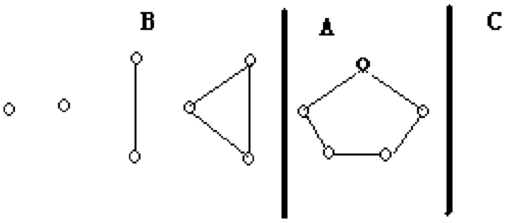


FIG.8

If(some connected component(G_i) has total vertex cost exceeding 2/3)

{

let $A^{'}$, $B^{'}$, $C^{'}$ be the resulting partition

let A------the set among $A^{'}$ and $B^{'}$ with greater cost

let B------the remaining vertices of G

let C-------$C^{'}$

Then A and B have cost not exceeding 2/3

So the theorem is true

}

for example: the FIG.9 can be divided into three sets A,B,C. see FIG.10

G_1 G_2 G_i

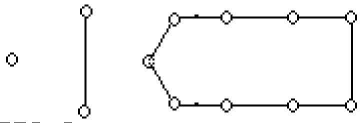


FIG. 9

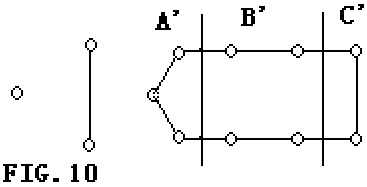


FIG. 10

A={B'}=1/3

B={ G_1, G_2, A'}=1/2

C={C'}=1/6

} // //end of if(! connected(G))

So it proves the theorem for all planar graphs. In all cases the separator C is either empty or contained in only one connected component of G.

COROLLARY 2($\sqrt{n}$ -separator theorem): *Let G be any n-vertex planar graph. The vertices of G can be partitioned into three sets A, B, C such that no edge joins a vertex in A with a vertex in B, neither A nor B contains more than 2n/3 vertices, and C contains no more than $2\sqrt{2}\sqrt{n}$ vertices.*

If the constant factor is 1/2 instead of 2/3, the constant factor of $2\sqrt{2}$ is allowed to increase.

COROLLARY 3: *Let G be any n -vertex planar graph having non -negative vertex costs summing to no more than one. Then the vertices of G can be partitioned into three sets A, B, C such that no edge joins a vertex in A with a vertex in B ,neither A nor B has total cost exceeding 1/2, and C contains no more than $2\sqrt{2}\sqrt{n}/(1-\sqrt{2/3})$ vertices.*

If graphs are almost planar, they also have a $\sqrt{n}$-separator theorem, so if we extend theorem 4, we can get the following theorem for those almost -planar graphs.

THEOREM 5: *Let G be an n-vertex finite element graph with nonnegative vertex costs summing to no more than* *one. Suppose mo element of G has more than k boundary vertices. Then the vertices of G can be partitioned into three sets A,B,C such that no edge joins a vertex in A with a vertex in B, neither A nor B has total cost exceeding 2/3, and C contains no more than* $4\ k/2\ \sqrt{n}$ *vertices.*

The following theorem shows that theorem 4 and its corollaries are tight to within a constant factor. That is, if f(n)=o(n), no f(n)-separator theorem holds for planar graph.

THEOREM 6: *For any k, let G=(V,E) be a k×k square grid graph(a k×k square section of the infinite grid graph). Let A be any subset of V such that αn≤|A|≤n/2,where n=k2 and α is a positive constant less than 1/2. The number of vertices in V-A adjacent to some vertex in A is at least k. . min{1/2,* $\sqrt{\alpha}$ *}*

An algorithm for finding a good partition

Step 1. Find a planar embedding of G and construct a representation for it（Figure 11）--**Time: O(n)**

We will use a list structure whose elements correspond to the edges of the graph. Stored with each edge are its endpoints and four pointers, designating the edges immediately clockwise and counter-clockwise around each of the endpoints of the edge.Stored with each vertex is some incident edge.

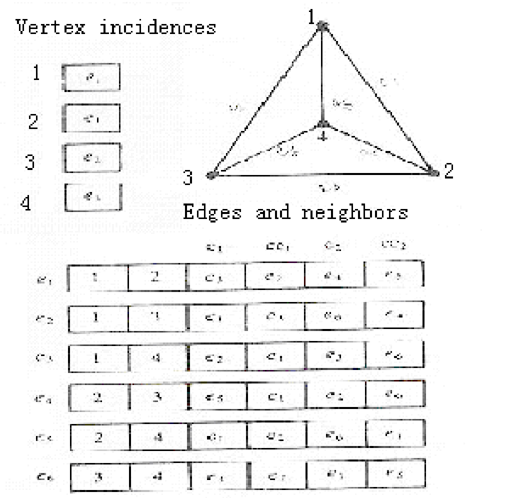


FIG. 11
Representation of an embedded planar graph

Step 2. Find the connected components of G and determine the cost of each one. If(none has cost exceeding 2/3)

{

construct the partition as described in the proof of Theorem 4.

}else

{

go to step 3.

}

Time:O(n)

Step 3.Find a Breadth-first spanning tree of the most costly component. Compute the level of each vertex and the number of vertices L(l) in each level l.

Time: O(n)

Step 4.Find the level l_1 such that the total cost of levels through l_1-1 does not exceed 1/2,but the total cost of levels 0 through l_1 does exceed 1/2. Let k be the number of vertices in levels 0 through l_1.

Time: O(n)

Step 5.Find the highest level $l_0 < l_1$ such that $L(l_0)+2(l_1- l_0) \leq 2\sqrt{k}$. Find the lowest level $l_2 \geq l_1+1$ such that $L(l_2)+2(l_2-l_1-1) \leq 2\sqrt{n-k}$.

Time: O(n)

Step 6:Delete all vertices on level l_2 and above. Construct a new vertex x to represent all vertices on levels 0 through l_0. Construct a Boolean table with one entry per vertex. Initialize to true the entry for each vertex on levels 0 through l_0. Initialize to false the entry for each vertex on levels l_0+1 through l_2-1. The vertices on levels 0 through l_0 correspond to a subtree of the breadth-first spanning tree generated in Step 3. Scan the edges incident to this tree clockwise around the tree. When scanning an edge(v,w) with v in the tree, check the table entry for w.

```
If(true)
{
   delete edge(v,w).
}else
{
   change it to true,
   construct an edge(x,w),
   delete edge(v,w).
}
```

the result of this step is a planar representation of the shrunken graph.

Time: O(n)

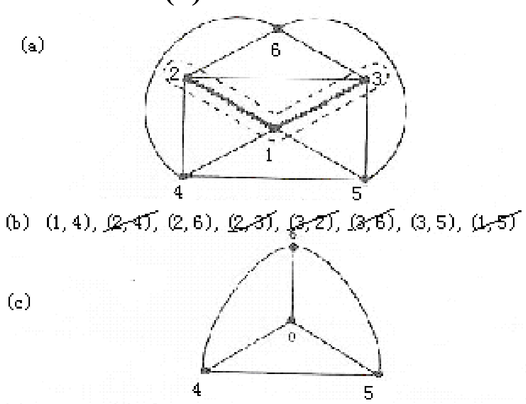


FIG. 12. Shrinking a subtree of a planar graph

(a) Original graph.

(b) Edges scanned around subtree.Those forming loops and multiple edges in shrunken graph are crossed out.

(c) Shrunken graph. Vertex 0 replace subtree.

Step7. Construct a breadth-first spanning tree rooted at x in the new graph.Record, for each vertex v, the parent of v in the tree, and the total cost of all descendants of v including v itself. Make all faces of the new graph into triangles by scanning the boundary of each face and adding(nontree) edges as necessary.

Time: O(n)

Step 8. Choose any nontree edge(v_1,w_1). Locate the corresponding cycle by following parent pointers from v_1 to w_1.Compute the cost on each side of this cycle by scanning the tree edges incident on either side of the cycle and summing their associated costs.

```
If((v,w) is a tree edge with v on the cycle and w not on the cycle)
{
     if(v is the parent of w)
     {
          the cost associated with (v, w) is the descendant cost of w.
     }
     if(w is the parent of v)
     {
          the cost of all vertices minus the descendant cost of v.
     }
}
if(which side of the cycle has greater cost)
{
        call it the "inside".
}
```

Time:O(n)

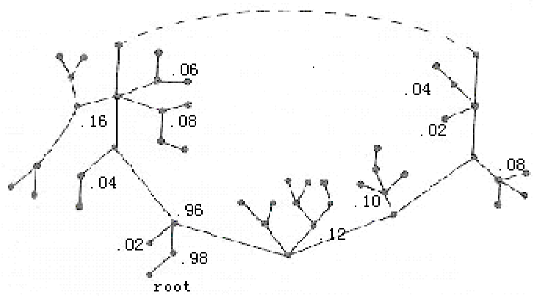


FIG13 Cycle constructed in Step 8. All vertices have cost.02. Numbers on vertices are descendant costs. The total cost inside the cycle is .48, outside the cycle is .34, and on the cycle is.18.

Step9.Let(v_i,w_i) be the nontree edge whose cycle is the current candidate to complete the separator. If the cost inside the cycle exceeds 2/3,need to find a better cycle by the following method.

Locate the triangle (v_i, y, w_i) which has(v_i,w_i) as a boundary edge and lies inside the (v_i,w_i) cycle. If either (v_i, y) or (y, w_i) is a tree edge, let (v_{i+1},w_{i+1}) be the nontree edge among (v_i,y) and (y, w_i).Compute the cost inside the (v_{i+1},w_{i+1}) cycle from the cost inside the (v_i,w_i) cycle and the cost of v_i, y and w_i.

If neither (v_i, y) nor (y,w_i) is a tree edge, determine the tree path from y to the (v_i,w_i) cycle by following parent pointers from y. Let z be the vertex on the (v_i,w_i) cycle reached during this seach. Compute the total cost of all vertices except z on this tree path. Scan the tree edges inside the (y,w_i) cycle, alternately scanning an edge in one cycle and an edge in the other cycle. Stop scanning when all edges inside one of the cycles have been scanned. Compute the cost inside this cycle by summing the associated cost of all scanned edges. Use the cost, the cost inside the (v_i, w_i) cycle, and the cost on the tree path from y to z compute the cost inside the other cycle. Let $(v_{i+1},w_{i+!})$ be the edge among (v_i, y) and (y,w_i) whose cycle has more cost inside it.

Repeat this step until finding a cycle whose inside has cost not exceeding 2/3.

Time:O(n)

Step10.Use the cycle found in step 9 and the levels found in step 4 to construct a satisfactory vertex partition as described in the proof of Lemma 3. Extend this partition from the connected component chosen in Step 2 to the entire graph as described in the proof of Theorem 4.

Time : O(n)

APPLICATIONS

- Generalized nested dissection
- Pebbling
- The post office problem
- Data structure embedding problem
- Lower bounds on Boolean circuits

CONCLUSION

This paper told us that the vertices of any n-vertex planar graph can be partitioned into three sets A, B, C such that no edge joins a vertex in A with a vertex in B, neither A nor B contains more than 2n/3 vertices, and C contains no more than $2\sqrt{2}\sqrt{n}$ vertices. Meanwhile, it exhibited an algorithm which found such a partition A, B, C in O(n) time.

REFERENCES

[1]R.J.LIPTON AND R.E.TARJAN *A Separator Theorem For Planar Graphs*. SIAM J.APPL. MATH.Vol.36,No 2,April 1979

[2]F.HARARY, *Graph Theory,* Addison - Wesley, Reading, MA, 1969