Carleton University - Summer 2018 Session

Please e-mail your answers to anil@scs.carleton.ca by **4:00 PM** today.

Your answers should have the following format:
Q1: <Your Answer>
Q2: <Your Answer>
Q3: <Your Answer>
....

Recall these following handy properties and then answer the questions in following pages.

- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
- Newton: $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.
- For $0 < x < 1$, $\sum_{n=0}^{\infty} x^n = 1/(1 - x)$.
- Geometric distribution: Assume an experiment has a success probability of p. We perform the experiment until it is successful for the first time. The expected number of times we perform the experiment is $1/p$.

1. How many bitstrings $s_1 s_2 \cdots s_{20}$ of length 20 have the property that $s_1 s_2 s_3 = 000$ or $s_2 s_3 s_4 = 000$?

   (a) $2^{17} - 2^{15}$

   (b) $2^{17} - 2^{16}$

   (c) $2^{18} - 2^{16}$

   (d) $2^{18} - 2^{17}$

2. What is the coefficient of $x^{15} y^5$ in the expansion of

$$(-3x + 5y)^{20}$$

   (a) $\binom{20}{5} \cdot 3^{15} \cdot 5^5$

   (b) $-\binom{20}{5} \cdot 3^{15} \cdot 5^5$

   (c) $\binom{20}{5} \cdot 5^{15} \cdot 3^5$

   (d) $-\binom{20}{5} \cdot 5^{15} \cdot 3^5$

3. A bowl contains 5 red balls and 7 blue balls. We choose a uniformly random subset of 3 balls. Define the event

$$A = \text{“exactly 2 of the chosen balls are red”}.$$

   What is $\Pr(A)$?

   (a) $\frac{\binom{5}{2}}{\binom{12}{3}}$

   (b) $\frac{5 \cdot \binom{7}{2}}{\binom{12}{3}}$

   (c) $\frac{\binom{5}{2} \cdot 7}{\binom{12}{3}}$

   (d) $\frac{\binom{7}{2}}{\binom{12}{3}}$

4. Let n be the number of students who are writing this exam. Each of these students has a uniformly random birthday, which is independent of the birthdays of the other students. We ignore leap years; thus, the year has 365 days. Define the event

$$A = \text{"at least one student's birthday is on December 21".}$$

What is $\Pr(A)$?

(a) $n \cdot (1/365) \cdot (364/365)^{n-1}$

(b) $365 \cdot n \cdot (364/365)^{n-1}$

(c) $1 - (1/365)^n$

(d) $1 - (364/365)^n$

5. The n students $S_1, S_2, \ldots, S_n$ decide to organize a Secret Gift Exchange party. They take a uniformly random permutation $P_1, P_2, \ldots, P_n$ of $S_1, S_2, \ldots, S_n$. For each $i = 1, 2, \ldots, n$, student S_i buys a gift and gives it, anonymously, to student P_i. Let X be the number of students who give their gift to themselves. What is the expected value $\mathbb{E}(X)$ of the random variable X?
Hint: Use an indicator random variable for each student.

(a) 1

(b) 2

(c) $1 + 1/n$

(d) $2 + 1/n$

6. You repeatedly, and independently, flip three fair coins, until there are exactly two heads among the three flips. Define the random variable X to be the total number of coin flips. For example, if the coin flips result in

$$(TTT), (THT), (HHH), (HTH),$$

then $X = 12$.
What is the expected value $\mathbb{E}(X)$ of X?

(a) 3/8

(b) 8/3

(c) 8

(d) 12

7. Let S be a uniformly random 2-element subset of $\{1, 2, 3, 4, 5, 6\}$, and let X be the number of elements of S that are even. What is the expected value $\mathbb{E}(X)$ of X?

   (a) 1
   (b) 3/2
   (c) 2
   (d) 5/2

8. How many bitstrings $s_1 s_2 \cdots s_{20}$ of length 20 have the property that $s_1 s_2 s_3 = 100$ or $s_2 s_3 s_4 = 001$?

   (a) $2^{17} - 2^{15}$
   (b) $2^{17} - 2^{16}$
   (c) $2^{18} - 2^{16}$
   (d) $2^{18} - 2^{17}$

9. Four friends each choose a uniformly random number from $\{1, 2, 3, 4, 5\}$. What is the probability that at least two of them chose the same number?

   (a) 1/5
   (b) 4/5
   (c) 24/125
   (d) 101/125

10. A standard deck of 52 cards contains four suits: $\spadesuit$, $\heartsuit$, $\clubsuit$, and $\diamondsuit$; there are 13 cards in each suit. Consider an experiment where one card, chosen uniformly at random, is discarded from the standard deck. (We do not know what is the suit of the discarded card.) Now we draw a card, uniformly at random, from this reduced deck of 51 cards. What is the probability that the drawn card is from $\spadesuit$?

    (a) 12/52
    (b) 13/51
    (c) 12/51
    (d) 1/4

11. Suppose you flip a fair coin five times. Let the random variable X count the total number of heads obtained in these flips. What is the probability that $X \geq 2$?
    (a) 1/3
    (b) 2/3
    (c) 3/16
    (d) 13/16

12. A grocery store has enough supply of 4 different types of fruits: apples, bananas, pears, and oranges. The storekeeper wants to make pre-packaged bags, so that each bag contains exactly 10 fruits, and each bag must contain all types of fruits. Formally, let B_i be one such bag and let $B_i = (a_i, b_i, p_i, o_i)$ represent the contents of B_i, where a_i is the numbers of apples, b_i is the number of bananas, p_i is the number of pears, and o_i is the number of oranges in B_i. Note that $a_i + b_i + p_i + o_i = 10$, and $a_i \geq 1$, $b_i \geq 1$, $p_i \geq 1$, and $o_i \geq 1$. The storekeeper wants to know from you, how many different types of bags she can form? (We say any two bags B_i and B_j are different if at least one of the following is true (a) $a_i \neq a_j$, (b) $b_i \neq b_j$, (c) $p_i \neq p_j$, or (d) $o_i \neq o_j$.)
    (a) 84
    (b) 120
    (c) 24
    (d) None of the above.

For the next three questions, suppose you throw 10 balls, uniformly at random, in 8 bins. Answer the following:

13. What is the probability that no ball lands in the 1st bin?
14. What is the probability that exactly 5 balls land in the 1st bin?
15. What is the probability that at least 5 balls land in the 1st bin?
16. Consider the a web graph on three nodes A, B, and C. Page A has hyperlinks to page B and C. Page B has hyperlink to A and Page C has hyperlinks to Pages A and B. Compute the Page Ranks of A, B, and C by following the process we outlined in the class today.
17. This problem is similar to the previous once, except that B has no hyperlinks, i.e. Page A has hyperlinks to B and C, and C has hyperlinks to A and B. Compute the Page Ranks of A, B, and C.