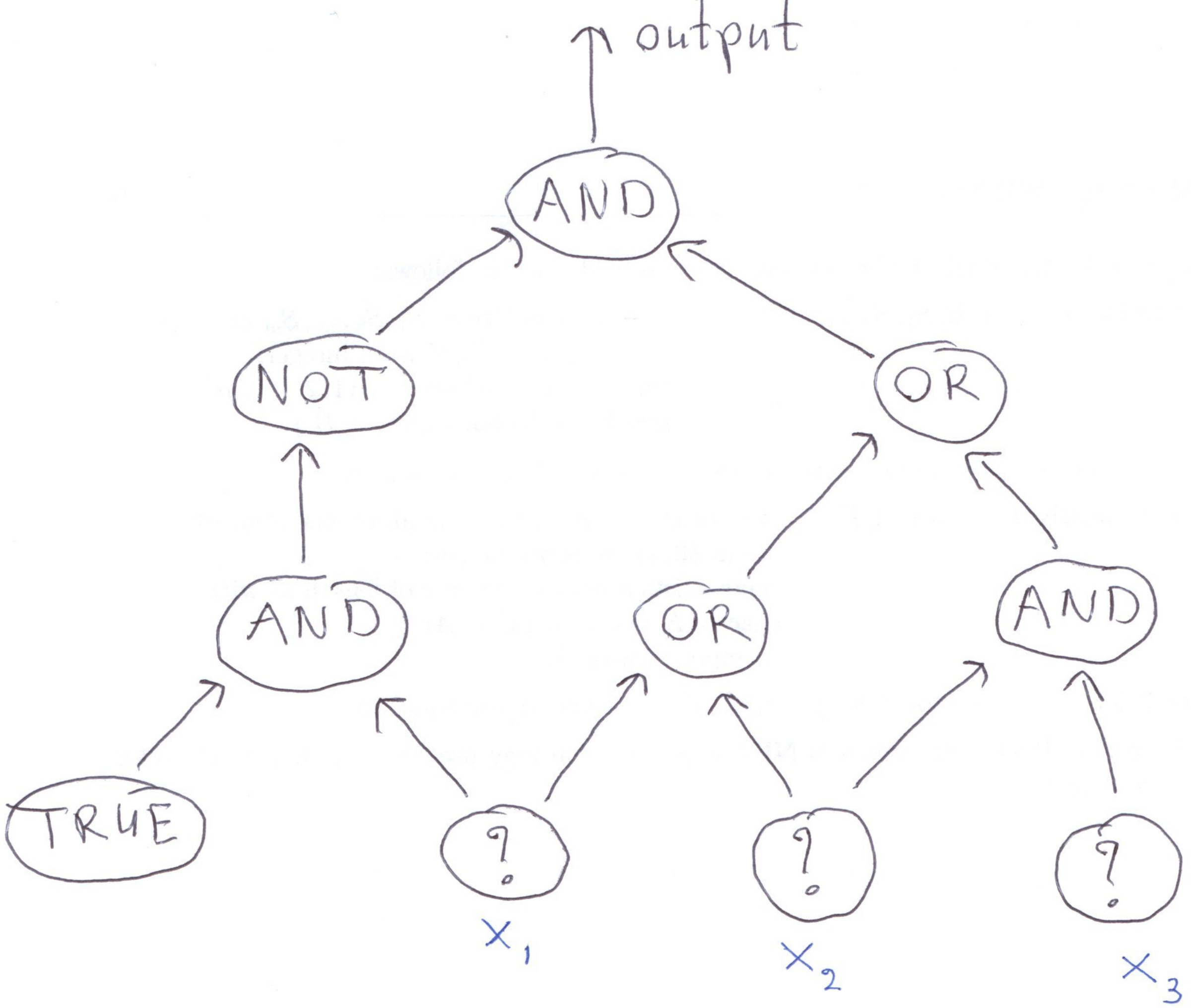
We will show that CIRCUIT-SAT is NP-complete.

Input: Boolean circuit

- directed acyclic graph, vertices are gates,
- AND-gates and OR-gates have indegree 2,
- NOT-gates have indegree 1,
- known input gates have indegree 0 and are labeled TRUE or FALSE,
- unknown input gates have indegree 0 and are labeled "?",
- there is one output gate (whose outdegree is 0)

Question: Is it possible to assign a truth-value to each unknown input gate, such that the output of the circuit is TRUE?



$x_1 = 1,\ x_2 = 0,\ x_3 = 1$: output $= 0$

$x_1 = 0,\ x_2 = 1,\ x_3 = 1$: output $= 1$

CIRCUIT-SAT

= { B : B is a Boolean circuit such that ∃ truth-values for the unknown input gates such that the output of B is TRUE }

To show that CIRCUIT-SAT is NP-complete, we have to do the following:

* Show that CIRCUIT-SAT is in NP:
  - certificate = sequence for truth-values for the unknown input gates
  - verification = evaluate the circuit (use topological sort)

* Show that for all $L \in NP$: $L \leq_P$ CIRCUIT-SAT.

Let $L \in NP$. We need a function f such that

① f: input x for $L \rightarrow$ Boolean circuit $B = f(x)$,

② $x \in L \iff B \in$ CIRCUIT-SAT,

③ time to compute B is polynomial in the length of x.

We know that $L \in NP$:

verification algorithm V

- input to V is (x,y) where x is an input for L and y is a certificate.

- $x \in L \iff$

  $\exists$ certificate y such that

  $|y| \leq |x|^c$,

  $V(x,y)$ returns YES,

  running time of $V(x,y)$ is $\leq |x|^{c'}$.

We now define the function f:

Let x be an input for L.

Define a new algorithm V_x:

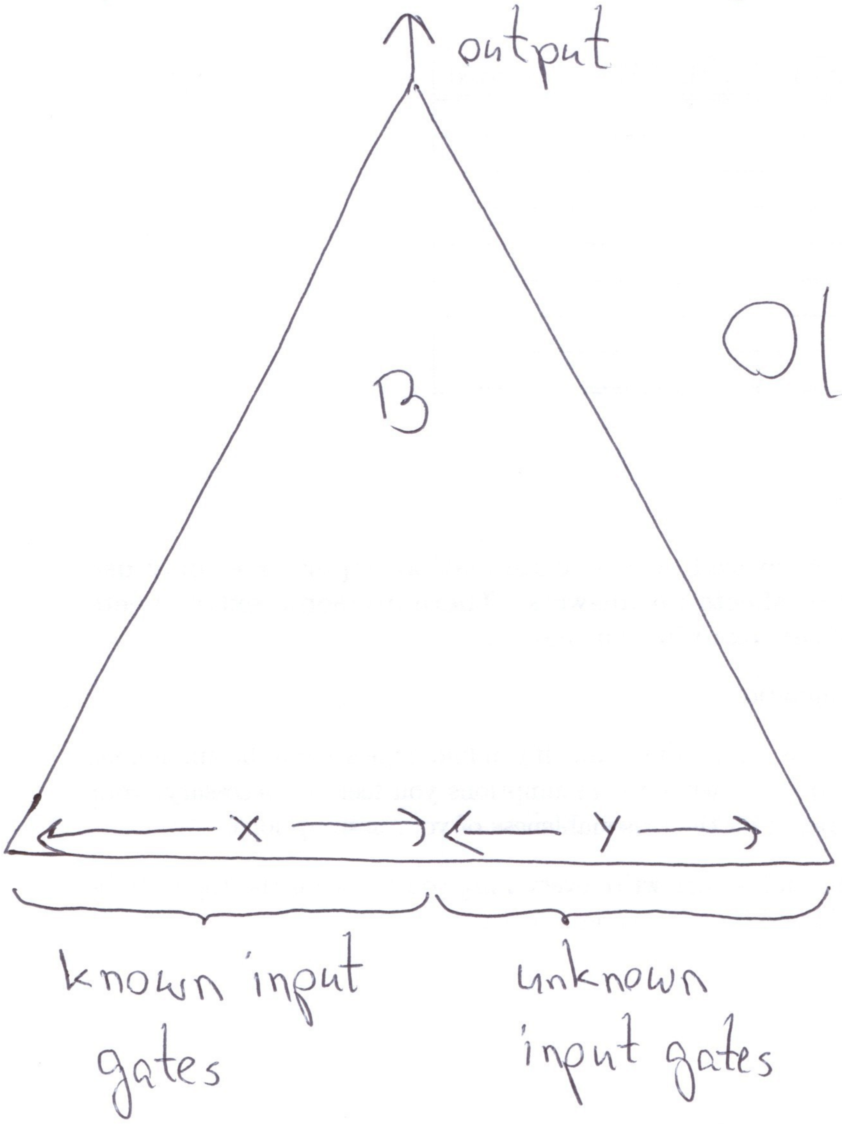
- input is a string y of length $\leq |x|^c$
- $V_x(y)$ runs $V(x,y)$.
- if $V(x,y)$ terminates in $\leq |x|^{c'}$ steps, then $V_x(y)$ terminates and returns the output of $V(x,y)$.
- if $V(x,y)$ has not terminated after $|x|^{c'}$ steps, then $V_x(y)$ terminates and returns NO.

Observe: * running time of algorithm V_x is $\leq |x|^{c'}$

* $x \in L \iff \exists$ input y for algorithm V_x such that $V_x(y)$ returns YES.

Algorithm V_x is a program that can be run on a computer

$\therefore$ V_x can be represented by a Boolean circuit B:



size of B:

$$O\left(|x| + \underbrace{|y|}_{\leq |x|^c} + |x|^{c'}\right)$$

polynomial in $|x|$.

The function f maps x to $\mathcal{B}$.

$$x \in L \iff \exists y : V_x(y) \text{ returns YES}$$

$$\iff \exists y : \text{output of } \mathcal{B} \text{ is TRUE}$$

$$\iff \mathcal{B} \in \text{CIRCUIT-SAT}.$$

<u>Conclusion:</u> CIRCUIT-SAT is NP-complete.

Now we can start using the theorem on page 206 to show that other problems are NP-complete.