Correctness of Dijkstra's algorithm:

Claim: for every vertex v:

* at the moment when $d(v)$ is minimum in Q, we have $d(v) = \delta(s,v)$.
* from that moment, $d(v)$ does not change any more.

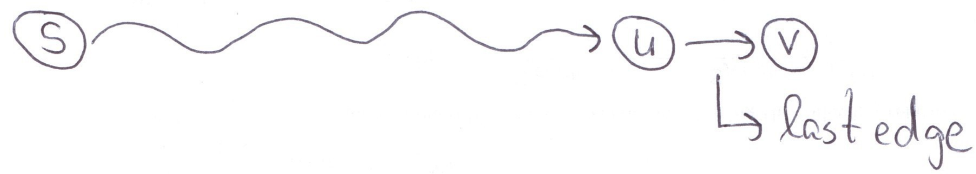
① At any moment: $\delta(s,v) \le d(v)$ for every vertex.

Proof: as on page 93. $\square$

② Assume at some moment, $d(v)$ becomes equal to $\delta(s,v)$. Then, during the rest of the algorithm, $d(v)$ does not change.

Proof: as on page 93. $\square$

3 Let $v \neq s$. Consider the shortest path from s to v.



Consider the iteration in which u is chosen as the vertex in Q whose d-value is minimum.

If $d(u) = \delta(s,u)$ at the beginning of this iteration, then $d(v) = \delta(s,v)$ at the end of this iteration.

<u>Proof</u>: As on page 94: At the end of this iteration,

$$
\begin{aligned}
d(v) &\leq d(u) + wt(u,v) && \text{// from algorithm} \\
&= \delta(s,u) + wt(u,v) && \text{// from assumption} \\
&= \delta(s,v) && \text{// property of shortest paths} \\
&\leq d(v) && \text{// from (1)}
\end{aligned}
$$

$\therefore\ d(v) = \delta(s,v)$.

$\square$

4 The minimum d-value in Q never decreases.

Proof: Consider an iteration of the while-loop, and let $u \in Q$ be such that $d(u)$ is minimum.

Before the for-loop: $d(u) \leq d(v)$ for all $v \in Q$.

During the for-loop, some values $d(v)$ may change.

If $d(v)$ is changed, its new value is

$$d(v) = d(u) + wt(u,v) > d(u).$$

$\therefore$ At the end of the for-loop: $d(u) \leq d(v)$ for all $v \in Q$.

$\therefore$ In the next iteration of the while-loop:

$$\text{all d-values in } \underbrace{Q \setminus \{u\}}_{\text{this is the new set } Q} \text{ are } \geq d(u)$$

$\therefore$ minimum d-value in Q^{new} is $\geq d(u)$.

$\square$

Proof of the claim on page 102:

First observe: in each iteration of the while-loop, the set Q gets smaller.

$\therefore$ for each vertex v, at some moment, $d(v)$ is minimum over all vertices in Q.

For $v = s$: the claim is true.

Consider a vertex v with $v \neq s$. Consider the shortest path P from s to v:

$$s \to u_1 \to u_2 \to u_3 \to \cdots \to u_k \to v$$

Observe: for each i with $1 \leq i \leq k$: the path
$s \to u_1 \to u_2 \to \ldots \to u_i$ is the shortest path from s to u_i.

Consider the first iteration of the while-loop:

Vertex s is chosen.

At that moment: $d(s) = 0 = \delta(s,s)$.

From 3: at the end of the iteration in which s is chosen as the vertex in Q with minimum d-value: $d(u_1) = \delta(s,u_1)$.

From 2: $d(u_1)$ does not change afterwards.

Observe: u_1 is still in Q at the end of the iteration in which s is chosen.

$\therefore$ At the beginning of the iteration in which u_1 is chosen: $d(u_1) = \delta(s,u_1)$.

From 3: At the end of the iteration in which u_1 is chosen: $d(u_2) = \delta(s,u_2)$.

From 2: $d(u_2)$ does not change afterwards.

Observe: $d(u_2) = \delta(s,u_2) = \delta(s,u_1) + wt(u_1,u_2)$

$$> \delta(s,u_1) = d(u_1)$$

From 4: At the end of the iteration in which u_1 is chosen: u_2 is still in Q.

$\therefore$ At the beginning of the iteration in which u_2 is chosen: $d(u_2) = \delta(s, u_2)$.

From 3: At the end of the iteration in which u_2 is chosen: $d(u_3) = \delta(s, u_3)$.

From 2: $d(u_3)$ does not change afterwards.

Observe: $d(u_3) = \delta(s, u_3) = \delta(s, u_2) + wt(u_2, u_3)$

$$> \delta(s, u_2) = d(u_2)$$

From 4: at the end of the iteration in which u_2 is chosen: u_3 is still in Q.

$\vdots$

At the beginning of the iteration in which u_k is chosen: $d(u_k) = \delta(s, u_k)$.

From 3: At the end of the iteration in which u_k is chosen: $d(v) = \delta(s, v)$.

From 2: $d(v)$ does not change afterwards.

Observe: $d(v) = \delta(s,v) = \delta(s,u_k) + wt(u_k,v)$

$$> \delta(s,u_k) = d(u_k)$$

From 4 : at the end of the iteration in which u_k is chosen : v is still in Q.

$\therefore$ ~~At the moment~~ At the beginning of the iteration in which v is chosen : $d(v) = \delta(s,v)$.

From 2 : $d(v)$ does not change afterwards.

$\square$