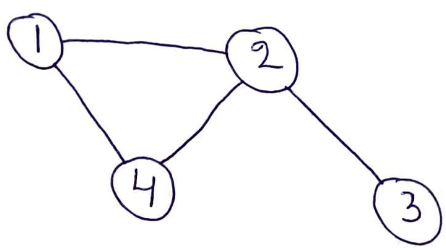
Graphs

What is a graph: $G = (V, E)$

V = set of vertices (nodes)

Undirected: E is a set of edges

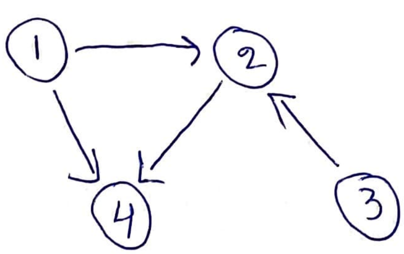
each edge is a pair $\{u, v\}$, where $u \in V$, $v \in V$, $u \neq v$.



$V = \{1, 2, 3, 4\}$

$E = \{ \{1,2\}, \{1,4\}, \{2,3\}, \{2,4\} \}$

directed: each edge is an ordered pair (u, v), where $u \in V$, $v \in V$, $u \neq v$ ("one-way street")



$V = \{1, 2, 3, 4\}$

$E = \{ (1,2), (1,4), (2,4), (3,2) \}$

Examples of graphs

* map (= network of roads)
* facebook : vertices $\longleftrightarrow$ users [end of 2020: 2.7 billion]

  edge $\{u,v\}$ $\longleftrightarrow$ u and v are friends
* WWW : vertices $\longleftrightarrow$ webpages

  edge (u,v) $\longleftrightarrow$ webpage u has a link to webpage v
* scheduling exams

  vertices $\longleftrightarrow$ courses taught this term

  edge $\{u,v\}$ $\longleftrightarrow$ $\exists$ student who takes both course u and course v

  ($\therefore$ exams for u and v cannot be scheduled at the same time)

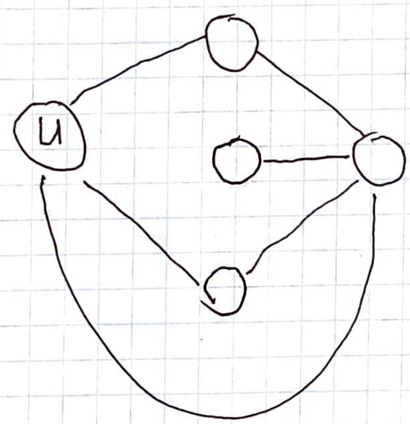
Let k be the smallest integer such that the following is possible:

① each vertex gets as label one element of $\{1,2,\dots,k\}$

② for each edge $\{u,v\}$: u and v have different labels

Then we can make an exam schedule with k time slots $t_1, t_2, \dots, t_k$:
all vertices (= courses) with label i have their exam in time slot t_i.
In this way there are no conflicts.
Note: Computing k is very difficult!

Undirected graph $G = (V, E)$.
degree(u) = # edges that contain u



degree$(u) = 3$

$$\sum_{u \in V} \text{degree}(u) = 2|E|.$$

How to store a graph?

$G = (V, E)$, $V = \{v_1, v_2, \dots, v_n\}$

* Adjacency matrix: $n \times n$ matrix

- if G is undirected:

$$\text{entry } (i,j) = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge} \\ 0 & \text{otherwise} \end{cases}$$

this gives a symmetric matrix

- if G is directed:

$$\text{entry } (i,j) = \begin{cases} 1 & \text{if } (v_i, v_j) \text{ is an edge} \\ 0 & \text{otherwise} \end{cases}$$

Advantage: in $O(1)$ time, we can test if there is an edge between two given vertices.

Disadvantage:

- uses $\Theta(n^2)$ space for any graph
- find all neighbors of a given vertex takes $\Theta(n)$ time.

* <u>Adjacency lists</u>: each vertex u stores a list.

- if G is undirected: the list of u stores all neighbors of u: all v for which $\{u,v\} \in E$
- if G is directed: the list of u stores all v for which $(u,v) \in E$ (outgoing edges)

Advantage:

- space $= \Theta(|V| + |E|)$
- all neighbors of vertex u can be found in $O(1 + \text{degree}(u))$ time.

Disadvantage: Testing if $\{u,v\}$ (or (u,v)) is an edge takes $O(1+\text{degree}(u))$ time.

For most algorithms: adjacency lists are the best choice.

Exploring an undirected graph $G=(V,E)$.

Given: vertex v.

Task: find all vertices that can be reached from v.

before the first call to explore

Algorithm explore(v): // initially, visited(u) = false for every vertex u

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visited(v) = true;
previsit(v);                     // see later
for each edge {v,u} ∈ E :
        if visited(u) = false: explore(u);
postvisit(v)                     // see later
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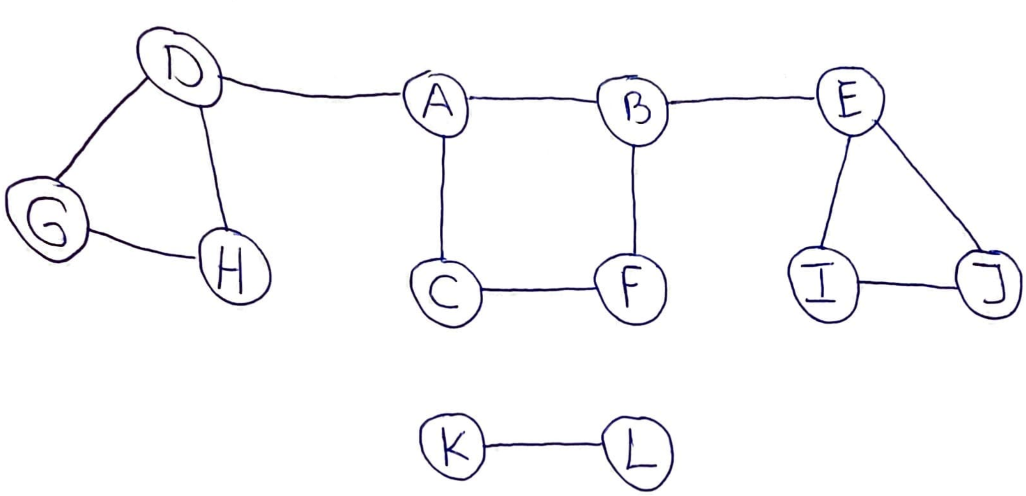
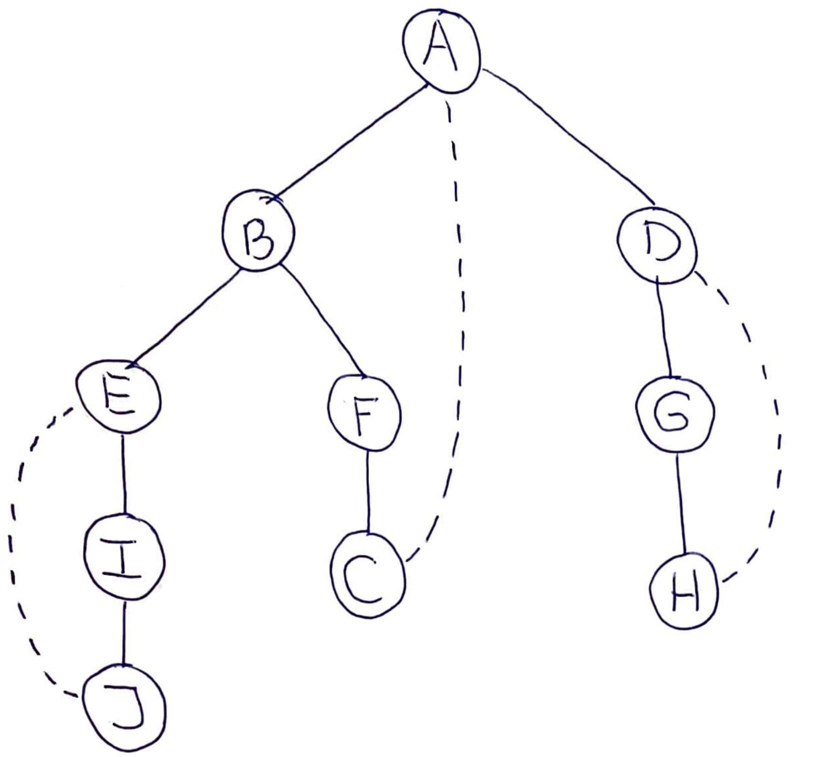


Run explore (A) ; in the for-loop : use alphabetical order (i.e., adjacency lists are sorted alphabetically).
Each time an edge $\{v,u\}$ is traversed (because visited (u) = false) : u is discovered for the first time;

draw $\{v,u\}$ as a solid edge.

all other edges : dotted.



solid edges form a tree (connected, no cycles)

these edges are called: <u>tree edges</u>

dotted edges: <u>back edges</u>

Why is algorithm explore(v) correct?

Why does it terminate: number of vertices u with visited(u) = false decreases in each recursive call.