Problems

Consult references [1][2].

Problem 1. Show that \(1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}\)

Problem 2. Show that \(1 + 2 + 3 + \cdots + n \in O(n^2)\)

Problem 3. Show that \(1 + 2 + 3 + \cdots + n \in \Omega(n^2)\)

Problem 4. Show that \(\sum_{k=1}^{n} k^2 \in \Theta(n^3)\)

Problem 5. Show that \(\frac{1}{3}n^2 - 3n \in O(n^2)\)

Problem 6. Let \(p(n) = \sum_{i=0}^{d} a_in^i\) be a polynomial of degree \(d\) and assume that \(a_d > 0\). Show that \(p(n) \in O(n^k)\), where \(k \geq d\) is a constant. What are \(c\) and \(n_0\) if we use the following definition of \(O\)-notation: \(O(g(n)) = \{f(n) : \exists c, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0\}\).

Problem 7. Let \(p(n) = \sum_{i=0}^{d} a_in^i\) be a polynomial of degree \(d\) and assume that \(a_d > 0\). For \(k \geq d\), show that \(\lim_{n \to \infty} \frac{\sum_{i=0}^{d} a_in^i}{n^k} = a_dn^{d-k} \geq a_d > 0\) and conclude that \(p(n) \in \Omega(n^k)\).

Problem 8. Let \(p(n) = \sum_{i=0}^{d} a_in^i\) be a polynomial of degree \(d\) and assume that \(a_d > 0\). Show that \(p(n) \in \Theta(n^d)\). (Hint: Using limits show that \(\lim_{n \to \infty} \frac{p(n)}{n^d} = a_d\) and \(0 < a_d < \infty\). Thus, \(p(n) \in \Theta(n^d)\). Recall that \(f(n) \in \Theta(g(n))\), if \(0 < \lim_{n \to \infty} \frac{f(n)}{g(n)} < \infty\).

Problem 9. Show that \(6n \log n + \sqrt{n} \log^2 n = \Theta(n \log n)\).

Problem 10. Fibonacci numbers are defined recursively as follows:
\(F_0 = 0, F_1 = 1,\) and \(F_n = F_{n-1} + F_{n-2}\) for any integer \(n > 1\).
Using induction show that \(F_n \geq 2^{n/2}\) for any \(n \geq 6\).

References
