### Problems

Consult [1, 2].

**Problem 1.** Let \( S = 1 + x + x^2 + \cdots + x^n \). Show the following

1. If \( x = 1 \), \( S = n + 1 \).
2. If \( x = 0 \), \( S = 1 \).
3. If \( x \neq \{0, 1\} \), \( S = \frac{1-x^{n+1}}{1-x} \).
4. If \( 0 < x < 1 \) and \( n \to \infty \), \( S = \frac{1}{1-x} \).
5. If \( 0 < x < 1 \), \( S = \frac{1-x^{n+1}}{1-x} \leq \frac{1}{1-x} = \Theta(1) \).
6. If \( x > 1 \), \( S_n = \frac{x^{n+1}-1}{x-1} \geq \frac{x^n-1}{x-1} = x^n \) and
   \( S_n = \frac{x^{n+1}-1}{x-1} = \frac{x}{x-1} x^n = O(x^n) \).
7. For \( x > 1 \), \( S = \Theta(x^n) \),
   i.e. \( S \) is proportional to the last term of the series.

**Problem 2.** Show that for positive constants \( \alpha, \beta \) and positive number \( n \), \( \alpha^{\log_\beta n} = n^{\log_\beta \alpha} \)

**Problem 3.** Consider the recurrence \( T(n) = aT\left(\frac{n}{2}\right) + cn^k \), where \( a \geq 1 \), \( b > 1 \), and \( c > 0 \) be constants.
Show that \( T(n) = O(n^{\log_b a}) + \sum_{i=0}^{\log_\beta n} a^i c \left( \frac{n}{b^i} \right)^k \).
Conclude that (i) if \( a > b^k \) then \( T(n) = \Theta(n^{\log_b a}) \),
(b) if \( a = b^k \) then \( T(n) = \Theta(n^{\log_\beta n}) \), and (c) if \( a < b^k \) then \( T(n) = \Theta(n^k) \).

**Problem 4.** Evaluate the recurrence \( T(n) = 2T\left(\frac{n}{2}\right) + n \), where \( T(1) = O(1) \).

**Problem 5.** Evaluate the recurrence \( T(n) = T\left(\frac{n}{2}\right) + 1 \), where \( T(1) = O(1) \).

**Problem 6.** Evaluate the recurrence \( T(n) = 3T\left(\frac{n}{2}\right) + n \), where \( T(1) = O(1) \).

**Problem 7.** Consider the recurrence \( T(n) = T(n/3) + T(2n/3) + n \). We can assume \( T(n) = O(1) \) for small values of \( n \). Show that \( T(n) = O(n\log n) \).

**Problem 8.** Evaluate the recurrence \( T(n) = 2T\left(\frac{n}{2}\right) + n\log n \), where \( T(1) = O(1) \). (Observe that it doesn’t fit any of the patterns in Problem 2.)

**Problem 9.** Let \( S \) be a set of \( n \) distinct real numbers. Devise an algorithm, running in \( O(n + k\log n) \) time, to report the \( k \) smallest elements of \( S \) in sorted order, where \( k \in \{1, \ldots, n\} \).

**Problem 10.** Let \( S \) be a set of \( n \) points on a real line. How fast can you find a pair of points that have the smallest distance? What if the points are in 2-dimensional real plane.
References
