Carleton University

Problem Set: 7

Problems

Consult [1, 2, 3]. These exercises are based on the complexity classes **P**, **NP**, and polynomial-time reducibility.

Problem 1. Formulate a decision problem corresponding to the following optimization problems:

- 1. (Clique) In an undirected graph G = (V, E), find the largest size clique. A set of vertices $K \subseteq V$ is said to form a clique, if for every pair of vertices $u, v \in K$, $uv \in E$.
- 2. (Independent Set) In an undirected graph G = (V, E), find the largest size independent set. A set of vertices $I \subseteq V$ is said to be independent, if for every pair of vertices $u, v \in I$, $uv \notin E$.
- 3. (Vertex Cover) In an undirected graph G = (V, E), find the smallest size vertex cover. A set of vertices $C \subseteq V$ is said to form a cover, if for every edge $e = (u, v) \in E$, $u \in C$ or $v \in C$.

Problem 2. For the above three decision problems, state an equivalent formulation in terms of the language of the decision problem.

Problem 3. Let k < n be a positive integer. Let us define the language $k-COMP = \{(G = (V, E), k) | G \text{ is a simple undirected graph on } n \text{ vertices containing at most } k \text{ components} \}.$ Is $k - COMP \in \mathbf{P}$? Is $k - COMP \in \mathbf{NP}$?

Problem 4. Are the decision version of the problems stated in Problem 1 in NP?

Problem 5. Let n = |V|. Show that the decision problems stated in Problem 1 are equivalent with respect to the polynomial time reducibility. I.e., show that

- 1. $Clique(G,k) \leq_P Independent-set(\bar{G},k)$
- 2. Independent-set(\bar{G}, k) \leq_P Vertex-Cover($\bar{G}, n-k$)
- 3. Vertex-Cover $(\bar{G}, n-k) \leq_P Clique(G, k)$

Problem 6. Show that each of the decision problems stated in Problem 1 are NP-complete.

Problem 7. Let S be a set of n positive numbers and let t be an integer value. Define the language Subset-Sum(S,t) as follows:

Subset-Sum(S,t) = {All sets S of n-numbers s.t. $\exists S' \subseteq S$, where $\sum_{x \in S'} x = t$ }.

Show that Subset-Sum(S,t) is **NP**-complete.

References

- T. Cormen, C. Leiserson, R. Rivest, and C. Stein, *Introduction to Algorithms*. 3rd. ed., MIT Press, 2009.
- [2] S. DasGupta, C. Papadimitriou, V. Vazirani. Introduction to Algorithms. McGraw Hill.
- [3] A. Maheshwari. *Notes on Algorithm Design*, Chapter 1, https://people.scs.carleton.ca/~maheshwa/Notes/DAA/notes.pdf