

# Clamshell Casting\*

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## Abstract

A popular manufacturing technique is *clamshell casting*, where liquid is poured into a cast and the cast is removed by a rotation once the liquid has hardened. We consider the case where the object to be manufactured is modeled by a polyhedron with combinatorial complexity  $n$  of arbitrary genus. The cast consists of exactly two parts and is removed by a rotation around a line in space. The following two problems are addressed: (1) Given a line of rotation  $l$  in space, we determine in  $O(n \log n)$  time whether there exists a partitioning of the cast into exactly two parts, such that one part can be rotated clockwise around  $l$  and the other part can be rotated counterclockwise around  $l$  without colliding with the interior of  $P$  or the cast. If the problem is restricted further, such a partitioning is only valid when no reflex edge or face of  $P$  is perpendicular to  $l$ , the algorithm runs in  $O(n)$  time. (2) An algorithm running in  $O(n^4 \log n)$  time is presented to find all the lines in space that allow a cast partitioning as described above. If we restrict the problem further and find all the lines in space that allow a cast partitioning as described above, such that no reflex edge or face of  $P$  is perpendicular to  $l$ , the algorithm's running time becomes  $O(n^4 \alpha(n))$ . All of the running times are shown to be almost optimal.

## 1 Introduction

The problem of whether a given object modeled by a polyhedron can be manufactured using the casting process is a well studied problem in computational geometry. The following overview of related problems is not extensive. For a detailed discussion of problems related to manufacturing processes considered in computational geometry, the reader is referred to Bose [6], Bose and Toussein [9], and the Handbook of Discrete and Computational Geometry by Goodman and O'Rourke [18].

We are concerned with the geometric setting of *clamshell casting* outlined below.

Assume that we wish to manufacture an object modeled by a polyhedron  $P$  with combinatorial complexity  $n$ . Let the boundary of  $P$  be the cast of  $P$ . Note that although this assumption is not realistic, the assumption allows us to study a simplified version of the problem to obtain more insight. Two problems are addressed. First, given a line of rotation  $l$  in space, we determine whether there exists a partitioning of the cast into exactly two parts, such that one part can be rotated clockwise around  $l$  and the other part can be rotated counterclockwise around  $l$  without colliding with the interior of  $P$  or the cast. We present an algorithm to solve this problem with running time

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$O(n \log n)$ . If we restrict the problem further and determine whether such a partitioning is possible when no reflex edge or face of  $P$  is perpendicular to  $l$ , the algorithm runs in  $O(n)$  time. Second, an algorithm is presented to find all the lines in space that allow a cast partitioning as described above. The algorithm's running time is  $O(n^4 \log n)$ . Again, we can restrict the problem further and find all the lines in space that allow a cast partitioning as described above, such that no reflex edge or face of  $P$  is perpendicular to  $l$ . The algorithm's running time is  $O(n^4 \alpha(n))$  in that case, where  $\alpha(n)$  is the inverse Ackermann function. Alternatively,  $P$  can be preprocessed in  $O(n^{4+\epsilon})$  expected time into a data structure of size  $O(n^{4+\epsilon})$ , such that for any given line  $l$ , we can decide in  $O(\log n)$  time if  $l$  is a valid casting line. We provide an example of a class of polyhedra with  $\Omega(n^4)$  combinatorially distinct valid casting lines. Hence, the algorithms we present are almost worst case optimal.

To our knowledge, this problem has not been considered in the past, but the problem was stated as an open problem in different publications [6, 7, 24]. The equivalent two-dimensional problem has been considered by Bose et al. [8]. Furthermore, the analogous three-dimensional translational casting problem, where the two parts of the cast are removed using a translation has been studied extensively in both two and three dimensions.

There exists a close relationship between rotational casting and translational casting. Let  $P$  denote a polyhedron and let  $l$  denote the axis of rotation. Assume that  $P$  and  $l$  are given in a cylindrical coordinate system with  $l$  as  $z$ -axis. Transform the coordinate system into a cartesian coordinate system, such that the  $x$ -axis describes the angle  $\phi$ , the  $y$ -axis describes the distance  $d$ , and the  $z$ -axis stays fixed. Considering this transformed system between  $x = 0$  and  $x = 2\pi$  shows the transformed cylindrical coordinate system. It is not hard to see that every point of the cast of  $P$  moves along a straight line when the cast is removed. This means that rotational casting becomes translational casting. It remains to analyze the shape of the transformed polyhedron  $\tilde{P}$ . Without loss of generality (since everything can be rotated), assume that  $P$  does not contain vertical edges or faces. Furthermore, assume that  $l$  does not intersect the interior of  $P$ , since otherwise,  $P$  could not be cast using  $l$  as axis of rotation. The reason is that there exists a point on the boundary of  $P$  that is the perpendicular projection of  $l$  on a face of  $P$ . This projection point cannot be rotated by an infinitesimal rotation around  $l$  without intersecting the interior of  $P$ . The transformation of a face  $f$  of  $P$  is in essence a curve describing the distance of points on the face to the origin. The distance from a face to a point is non-linear and can not be described using an algebraic surface, but trigonometric functions are necessary [5]. Since  $l$  does not intersect the interior of  $P$ , the transformed polyhedron  $\tilde{P}$  is topologically equivalent to  $P$  and its boundary consists of piecewise non-algebraic surface patches. considering rotational casting of  $P$  about the line  $l$  is equivalent to considering translational casting  $\tilde{P}$  with cast removal directions  $+x$  and  $-x$ .

To our knowledge, this close relationship between rotational casting and translational casting has not previously been mentioned or used to obtain algorithms to rotationally cast polygons or polyhedra. Although the transformation may be useful in testing whether a given line allows to cast  $P$ , we can not find a line in space that allows to cast  $P$  using this transformation only. Furthermore, none of the algorithms surveyed in the following can easily be extended to handle translational castability of 3-dimensional objects bounded by piecewise non-algebraic surface patches.

Different approaches exist to examine the three-dimensional version of the casting problem, where the object to be manufactured is modeled by a polyhedron in space with combinatorial complexity  $n$  and the polyhedron's boundary is used as cast and removed by translations. Ahn et al. [3] determine whether the cast can be partitioned into exactly two pieces, such that both pieces can be removed from the manufactured object by translations in opposite directions without

breaking the object or the cast. They consider two problems: the first problem is to decide whether a given object is castable with respect to a given cast removal direction and the second problem is to find a valid removal direction for a given object in case that one exists. Ahn et al. give sufficient and necessary conditions for translational castability of objects bounded by algebraic surface patches that meet along algebraic curves. However, algorithms to determine castability only work for polyhedra. For a polyhedron of combinatorial complexity  $n$ , they present algorithms to solve the first problem in time  $O(n \log n)$  and to solve the second problem in expected time  $O(n^4)$ . They also prove that their algorithm can not be extended to allow arbitrary cast removal directions, i.e. non-opposite directions. Ahn et al. [1] present necessary and sufficient conditions to test whether an object can be manufactured via translational casting using non-opposite removal directions. For a given pair of directions, the test can be performed in  $O(n^2 \log n)$  time. Before Ahn et al. [3] found an analytic solution to this problem, Hui and Tan [20] developed a heuristic method to solve the problem. They choose candidate removal directions heuristically and test for a sample of points located on the boundary of the polyhedron whether the points can be removed in that direction. Ahn et al. [2] solve the two above-mentioned problems using randomized algorithms in a setting, where the direction for cast removal is uncertain. This setting is important in practice, since the control of the casting machinery is imperfect.

Note that the 3-dimensional problems considered here are closely related to the problems considered by Ahn et al. in [3]. The 3-dimensional decision problem for translational casting considered by Ahn et al. corresponds to the 3-dimensional decision problem addressed in this paper. The problem of determining all the valid removal directions considered by Ahn et al. corresponds to the problem of determining all valid casting lines in space addressed in this paper. Since Ahn et al. propose algorithms that handle opposite cast removal only, the space of solutions is the space of all possible directions, i.e. the unit sphere of directions, and therefore 2-dimensional. The problem we address considers full lines. Hence the space of all solutions is the space of all possible lines in 3D and therefore 4-dimensional. Note that despite these two additional degrees of freedom, the algorithms we present are only slightly less efficient than the algorithms proposed by Ahn et al.

Bose et al. [7] consider an object modeled by a simple polyhedron with  $n$  vertices and use the polyhedron's boundary as cast. They determine whether the cast can be partitioned into two pieces by a plane, such that both pieces can be removed from the manufactured object by translation without breaking the object or the cast. If this is the case, the object can be manufactured by sand casting. A simple algorithm to decide whether a simple polyhedron with  $n$  vertices can be manufactured by sand casting running in time  $O(n^2 \log n)$  is provided. The running time of the algorithm can be improved to  $O(n^{\frac{3}{2}+\epsilon})$ , for any fixed  $\epsilon > 0$ , by using complicated data structures. Furthermore, the running time of the algorithm becomes  $O(n^2)$  in case that the cast removal directions are opposite directions.

Section 2 introduces the notation and preliminaries used throughout this paper. Section 3 discusses the problem of finding a partitioning of a given cast based on a given line of rotation, and Section 4 discusses the problem of finding all of the combinatorially distinct lines in space that allow a valid partitioning of the cast. Finally, Section 5 concludes and gives ideas for future work.

## 2 Preliminaries

Define a polyhedron  $P$  of arbitrary genus as a closed, compact, connected subset of  $\mathbb{R}^3$  bounded by a piecewise linear surface. Let  $\text{int}(P)$  and  $\partial P$  denote the interior and boundary of  $P$ , respectively,

so that  $P = \text{int}(P) \cup \partial P$ . The boundary is also called the *cast* of  $P$ . Two faces are *adjacent* if they share at least one edge. Parallel adjacent faces are not allowed, since this can be easily avoided by merging the two adjacent parallel faces. Let  $n$  denote the combinatorial complexity of  $P$ . The aim is to rotationally remove the cast of  $P$  in two pieces. We specify below precisely what this means.

**Definition 1.** Let  $l$  be a directed line in 3-dimensional space. Consider the plane  $\pi$  perpendicular to  $l$  passing through a point  $p$  of  $P$ . Denote the intersection point of  $\pi$  and  $l$  by  $l'(p)$ . Denote the circular arc with center  $l'(p)$  and angle  $\alpha$  starting at  $p$  winding in clockwise (cw) or counterclockwise (ccw) direction by  $\text{cwarc}(l'(p), p, \alpha)$  or  $\text{ccwarc}(l'(p), p, \alpha)$  respectively. A face  $f$  of  $P$  is called *removable in cw orientation with respect to  $l$*  if

$$\exists \alpha > 0 \text{ such that } \forall p \in \text{int}(f) : \text{cwarc}(l'(p), p, \alpha) \cap \text{int}(P) = \emptyset$$

and *removable in ccw orientation with respect to  $l$*  if

$$\exists \alpha > 0 \text{ such that } \forall p \in \text{int}(f) : \text{ccwarc}(l'(p), p, \alpha) \cap \text{int}(P) = \emptyset.$$

The cw and ccw orientation is measured with respect to the orientation of line  $l$ .

The cw or ccw orientation is then called a *valid orientation for cast removal for  $f$  with respect to  $l$*  respectively, and  $l$  is called a *valid casting line for  $f$* .

**Definition 2.** Let  $l$  be a directed line in 3-dimensional space. A polyhedron  $P$  is *rotationally castable with respect to  $l$* , if  $\partial P$  can be partitioned into exactly two non-empty connected components  $P_1$  and  $P_2$ , such that all faces of  $P_1$  are removable in cw orientation with respect to  $l$ , all faces of  $P_2$  are removable in ccw orientation with respect to  $l$ ,  $\exists \alpha > 0$  such that  $\forall p \in P_1$

$$\text{cwarc}(l'(p), p, \alpha) \cap \text{int}(P_2) = \emptyset,$$

and  $\exists \alpha > 0$  such that  $\forall p \in P_2$

$$\text{ccwarc}(l'(p), p, \alpha) \cap \text{int}(P_1) = \emptyset.$$

The last two conditions of the definition ensure that the two components  $P_1$  and  $P_2$  (which partition the boundary of  $P$ ) do not obstruct each other's rotational paths. Figure 1 shows part of a polyhedron that is decomposed into  $P_1$  and  $P_2$ . Although  $\partial P$  can be decomposed into  $P_1$  and  $P_2$ , it is impossible to split the face perpendicular to  $l$  in a way that allows the removal of the cast without intersecting  $\text{int}(P)$  and without the pieces  $P_1$  and  $P_2$  intersecting each other. This is despite the fact that each face individually is removable with respect to  $l$ . In the following, the notations rotationally castable and castable are used interchangeably. Note that the partitioning of the polyhedron is not necessarily at edges of  $P$ .

Let  $p \in \partial P$  be a point incident to  $k$  reflex edges  $e_i, 1 \leq i \leq k$  and  $s$  faces  $f_i, 1 \leq i \leq s$ . Denote the direction of  $e_i$  by  $\vec{d}(e_i)$  and the inner normal of  $f_i$  by  $\vec{n}(f_i)$ .

**Definition 3.** Let  $l$  be a directed line in 3-dimensional space in direction  $\vec{l}$ . A polyhedron  $P$  is *robustly castable with respect to  $l$* , if  $P$  is rotationally castable with respect to  $l$ ,

$$\forall p \in \partial P : \vec{d}(e_i) \cdot \vec{l} \neq 0, 1 \leq i \leq k, \text{ and}$$

$$\forall p \in \partial P : \vec{n}(f_i) \times \vec{l} \neq \mathbf{0}, 1 \leq j \leq s,$$

where  $\cdot$  denotes the dot product of two vectors and  $\times$  denotes the vector product of two vectors.

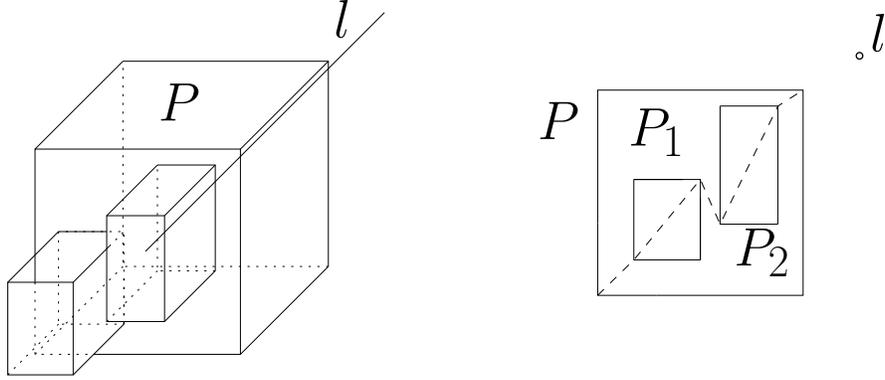


Figure 1: Two different views of  $P$  are shown along with the partitioning of  $P$ . The components  $P_1$  and  $P_2$  obstruct each other's rotational paths.

The last two conditions of the definition ensure that no face or reflex edge of  $P$  is perpendicular to  $l$ .

If  $P$  is robustly castable with respect to  $l$ , we say  $l$  is a valid robust casting line for  $P$ . The definition implies that  $P$  is robustly castable with respect to a line  $l$  if  $P$  is castable with respect to  $l$  and no reflex edge or face of  $P$  is perpendicular to  $l$ . However, there can still exist a point of  $\partial P$  that slides along the boundary of  $P$  when the cast parts are removed. Hence, the definition does not correspond to the intuitive definition of robust, where the limited precision of the casting machinery used to manufacture an object is taken into account. Therefore, the definition of robust castability does not imply that surface defects are less likely to occur in robust casting than in general casting. Although the definition of robust castability does not offer this advantage, it ensures that algorithms to determine robust castability of a polyhedron are more efficient than algorithms to determine general castability of a polyhedron.

The following sections discuss both the robust casting process and the general casting process.

### 3 Decision Problem

In this section, we address the problem of determining whether a polyhedron  $P$  with combinatorial complexity  $n$  is castable with respect to a given line of rotation and present an algorithm that solves the problem in  $O(n \log n)$  time. In case we want to determine if  $P$  is robustly castable, the algorithm's running time becomes  $O(n)$ .

Let  $P$  be a polyhedron and let  $l$  be a line in space. The problem can be decomposed into three subproblems: determining the valid orientation for cast removal for all faces of  $P$  with respect to  $l$ , checking whether all the faces removable in cw and ccw orientation with respect to  $l$  form a connected component of  $P$ , respectively, and testing whether the two components can be rotated by a small amount around  $l$  without colliding.

#### 3.1 Robust rotational casting

This section presents an algorithm to test whether a polyhedron  $P$  is robustly castable with respect to a line  $l$  in space in  $O(n)$  time.

**Definition 4.** Let  $f$  be a face of  $P$  and denote its inner normal by  $\vec{n}$ . Define the open unbounded prism  $S(f) = \{\vec{p} + t\vec{n} : p \in f \text{ and } t \in \mathbb{R}\}$ . Denote  $S^+(f) = \{\vec{p} + t\vec{n} : p \in f \text{ and } t > 0\}$  and  $S^-(f) = \{\vec{p} + t\vec{n} : p \in f \text{ and } t \leq 0\}$ , see Figure 2.

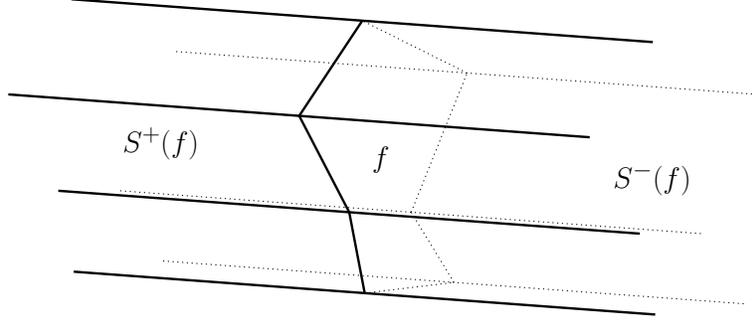


Figure 2: The prisms  $S^+(f)$  and  $S^-(f)$ .

In a first step, the algorithm determines whether any reflex edge or face of  $P$  is perpendicular to the given line  $l$ . This test takes constant time per reflex edge or face, respectively, and therefore  $O(n)$  total time. If any reflex edge or face of  $P$  is perpendicular to  $l$ ,  $P$  is not robustly castable with respect to  $l$ , by definition. Otherwise, we determine the valid orientation for cast removal for each face of  $P$ . In the following, we can therefore assume that no reflex edge or face of  $P$  is perpendicular to  $l$ .

The following lemma characterizes all locations from which a face is removable.

Let  $f$  be a face of  $P$  with inner normal  $\vec{n}$  and let  $l$  be a line in direction  $\vec{l}$  with  $\vec{n} \times \vec{l} \neq \mathbf{0}$ . If  $l \cap S(f) = \emptyset$ , compute a point  $p \in \partial f$  that minimizes the Euclidean distance between  $\partial f$  and  $l$ . If  $p$  is not unique, an arbitrary point  $p$  of that set is picked. Denote by  $\tau$  the plane with normal  $\vec{n} \times \vec{l}$  passing through  $p$  and by  $p^*$  the point  $p$  translated by  $\vec{n} \times \vec{l}$ . Denote the open half space induced by  $\tau$  containing  $p^*$  by  $\tau^+$  and the open half space induced by  $\tau$  not containing  $p^*$  by  $\tau^-$ . The following lemma characterizes all locations from which a face is removable.

**Lemma 1.** For the orientation for cast removal of  $f$ , the following four cases are possible:

1. The face  $f$  is removable from the cast using only a cw orientation around  $l$ , if and only if  $l \cap S(f) = \emptyset$  and  $l \in cl(\tau^-)$ .
2. The face  $f$  is removable from the cast using only a ccw orientation around  $l$ , if and only if  $l \cap S(f) = \emptyset$  and  $l \in cl(\tau^+)$ .
3. The face  $f$  needs to be partitioned into two or more parts along the orthogonal projection of  $l$  on  $f$ , if and only if  $l \cap S^+(f) = \emptyset$  and  $l \cap S^-(f) \neq \emptyset$ . One or more parts of  $f$  are removable using a ccw rotation and the other ones using a cw rotation around  $l$ .
4. The face  $f$  is not removable from the cast, if and only if  $l \cap S^+(f) \neq \emptyset$ .

*Proof.* Consider the plane  $\pi$  perpendicular to  $l$  passing through a point  $p$  of  $P$ . Denote the intersection point of  $\pi$  and  $l$  by  $l'(p)$ . Every point  $p \in int(f)$  moves on  $cwarc(l'(p), p, \alpha)$  or  $ccwarc(l'(p), p, \alpha)$  when rotated by an angle  $\alpha$  around  $l$ . Denote the vector from  $p$  to  $l'(p)$  by  $p\vec{l}'(p)$  and the vector

$\vec{l} \times \vec{p}l'(p)$  by  $\vec{p}l'(p)^\perp$ . The tangent of  $cwarc(l'(p), p, \alpha)$  or  $ccwarc(l'(p), p, \alpha)$  is  $\vec{p}l'(p)^\perp$  or  $-\vec{p}l'(p)^\perp$ , respectively, for any point  $p$ . Denote by  $\nu$  the plane passing through  $p$  with normal vector  $\vec{p}l'(p)$  and denote by  $\nu^+$  the open half space bounded by  $\nu$  containing  $l'$ . The two arcs  $cwarc(l'(p), p, \alpha)$  and  $ccwarc(l'(p), p, \alpha)$  are contained in  $cl(\nu^+)$ .

Let  $p$  be an arbitrary point in the interior of  $f$ . There exists an open ball  $b$  with positive radius centered at  $p$  with the property that exactly half of  $b$  is contained in  $int(P)$  and exactly half of  $b$  is contained in the exterior of  $P$ . Denote the ray starting at  $p$  propagating in direction  $\vec{p}l'(p)^\perp$  by  $q^+$  and denote the ray starting at  $p$  propagating in direction  $-\vec{p}l'(p)^\perp$  by  $q^-$ .

Assume that  $l \cap S(f) = \emptyset$  and  $l \in cl(\tau^-)$ , and let  $p$  be an arbitrary point in the interior of  $f$ . The intersection  $b \cap q^+$  is located completely outside of  $int(P)$ . Hence,  $p$  can move by a small amount along  $\vec{p}l'(p)^\perp$  without penetrating  $int(P)$ . Since  $cwarc(l'(p), p, \alpha) \subseteq cl(\nu^+)$  and since  $\vec{p}l'(p)^\perp$  is the tangent of  $cwarc(l'(p), p, \alpha)$  in  $p$ , small movements of  $p$  along  $cwarc(l'(p), p, \alpha)$  are possible without penetrating  $int(P)$ . Hence,  $\exists \alpha > 0$  such that  $\forall p \in int(f) : cwarc(l'(p), p, \alpha) \cap int(P) = \emptyset$ . The intersection  $b \cap q^-$  is completely contained in  $int(P) \cup \{p\}$  and hence,  $p$  cannot move infinitesimally along  $-\vec{p}l'(p)^\perp$  without penetrating  $int(P)$ . Since infinitesimal movements along  $-\vec{p}l'(p)^\perp$  correspond to infinitesimal movements along  $ccwarc(l'(p), p, \alpha)$ , there is no  $\alpha > 0$  such that  $\forall p \in int(f) : ccwarc(l'(p), p, \alpha) \cap int(P) = \emptyset$ . Hence,  $f$  is only removable using a cw orientation around  $l$ .

Assume that  $l \cap S(f) = \emptyset$  and  $l \in cl(\tau^+)$ , and let  $p$  be an arbitrary point in the interior of  $f$ . The intersection  $b \cap q^-$  is located completely outside of  $int(P)$ . Hence,  $p$  can move by a small amount along  $-\vec{p}l'(p)^\perp$  without penetrating  $int(P)$ . Since  $ccwarc(l'(p), p, \alpha) \subseteq g^+(p)$  and since  $-\vec{p}l'(p)^\perp$  is the tangent of  $ccwarc(l'(p), p, \alpha)$  in  $p$ , small movements of  $p$  along  $ccwarc(l'(p), p, \alpha)$  are possible without penetrating  $int(P)$ . Hence,  $\exists \alpha > 0$  such that  $\forall p \in int(f) : ccwarc(l'(p), p, \alpha) \cap int(P) = \emptyset$ . The intersection  $b \cap q^+$  is completely contained in  $int(P) \cup \{p\}$  and hence,  $p$  cannot move infinitesimally along  $\vec{p}l'(p)^\perp$  without penetrating  $int(P)$ . Since infinitesimal movements along  $\vec{p}l'(p)^\perp$  correspond to infinitesimal movements along  $cwarc(l'(p), p, \alpha)$ , there is no  $\alpha > 0$  such that  $\forall p \in int(f) : cwarc(l'(p), p, \alpha) \cap int(P) = \emptyset$ . Hence,  $f$  is only removable using a ccw orientation around  $l$ .

If  $l \cap S^+(f) = \emptyset$  and  $l \cap S^-(f) \neq \emptyset$ ,  $f$  is divided into two or more faces along the perpendicular projection  $l'$  of  $l$  on  $f$  by inserting one or more edges  $e$ . Denote the set of faces contained in  $cl(\tau^-)$  by  $F_1$  and the set of faces contained in  $cl(\tau^+)$  by  $F_2$ . For arbitrary  $f_1 \in F_1$  and  $f_2 \in F_2$ , the points on  $e$  globally minimize the distance between  $\partial f_1, \partial f_2$  and  $l$  respectively. Hence,  $f_1$  is removable using a cw orientation around  $l$  and  $f_2$  is removable using a ccw orientation around  $l$ . Note that  $F_1$  and  $F_2$  contain exactly one face each for every convex face  $f$ .

If  $l \cap S^+(f) \neq \emptyset$ , consider an arbitrary point  $p$  located on the orthogonal projection of  $l$  on  $f$ . The point  $p$  cannot be rotationally removed from the cast, i.e. there is no  $\alpha > 0$  such that  $cwarc(l'(p), p, \alpha) \cap int(P) = \emptyset$  or  $ccwarc(l'(p), p, \alpha) \cap int(P) = \emptyset$ . Therefore,  $f$  is not removable with respect to  $l$ .

This determines the removability of  $f$  depending on the location of  $l$  in 3-dimensional space. Hence, the four statements of Lemma 1 follow directly.  $\square$

Lemma 1 classifies all faces of  $P$  into four categories. In linear time, all the faces of  $P$  can be categorized. If any face of  $P$  is not removable with respect to  $l$ ,  $P$  is not castable with respect to  $l$ . Otherwise, the aim is to determine if the boundary of  $P$  can be decomposed into two connected components. In particular, the faces of  $P$  that need to be split (faces of type 3 in Lemma 1) need to be examined more carefully as outlined in the following lemma.

**Lemma 2.** *Let  $P$  be a polyhedron with combinatorial complexity  $n$  and let  $l$  be a line in space. All the faces  $f$  of  $P$  with  $l \cap S^+(f) = \emptyset$  and  $l \cap S^-(f) \neq \emptyset$  can be partitioned along the orthogonal projection of  $l$  on  $f$  and the resulting new faces can be labeled as removable in cw or ccw orientation with respect to  $l$  respectively in  $O(n)$  time.*

*Proof.* Consider a single face  $f$  with  $l \cap S^+(f) = \emptyset$  and  $l \cap S^-(f) \neq \emptyset$ . Lemma 1 states that  $f$  needs to be partitioned along the orthogonal projection of  $l$  on  $f$ . If we denote the number of vertices of  $f$  by  $r$ ,  $f$  is partitioned into  $O(r)$  faces, because every edge of  $f$  is split at most once. Assume that  $f$  is stored using a doubly-connected edge list. The  $s \leq r$  points on  $\partial f$ , where  $f$  needs to be split can be found ordered by their appearance on  $f$  in  $O(r)$  time by traversing the edges of  $f$  [17, Chapter 2]. Using the algorithm by Hoffmann et al. [19], it is possible to order the points with respect to their appearance on the projected line in time  $O(r)$ . Denote the points on  $\partial f$ , where  $f$  needs to be split ordered with respect to their appearance on the projected line by  $q_1, \dots, q_s$ . Note that  $s$  is even, because every edge that is added to  $f$  has two endpoints and no two new edges share a vertex. Adding the edges with endpoints  $q_1$  and  $q_2$ ,  $q_3$  and  $q_4$ ,  $\dots$ ,  $q_{s-1}$  and  $q_s$  partitions  $f$  along the orthogonal projection of  $l$  on  $f$ . Once the new edges and their intersection points with  $f$  are known, it takes time  $O(m)$  to insert the new edge into the doubly-connected edge list, where  $m$  is the number of edges adjacent to the new edge [17, Chapter 2]. In the worst case,  $O(r) = O(n)$  and  $O(m) = O(n)$ .

Next, we analyze the running time for splitting all of the faces  $f$  with  $l \cap S^+(f) = \emptyset$  and  $l \cap S^-(f) \neq \emptyset$  of  $P$ . Since every edge  $e$  of  $P$  has two adjacent faces,  $e$  needs to be split at most twice. This ensures that the total number of new edges is  $O(n)$ . Also, the total number of edges adjacent to the new edges is  $2n = O(n)$ . The reason for this is that each original edge of  $P$  is adjacent to at most two new edges. Hence, all of the faces can be split in  $O(n)$  total time. Each of the newly created faces can be labeled as removable in cw or ccw orientation with respect to  $l$ , respectively, in constant time. Therefore, it takes  $O(n)$  time to partition all the faces of  $P$  with  $l \cap S^+(f) = \emptyset$  and  $l \cap S^-(f) \neq \emptyset$  and to find the valid orientation with respect to  $l$  for each of the newly created faces.  $\square$

This yields the conclusion that all the faces of a polyhedron  $P$  with combinatorial complexity  $n$  can be labeled as removable in cw, ccw, or no orientation with respect to a line of rotation  $l$  in  $O(n)$  total time.

The next step is to determine if those labels on the faces admit a decomposition of  $\partial P$  into two connected sets. The following lemma allows this test to be done in linear time:

**Lemma 3.** *Given a partitioning of  $\partial P$  into two sets  $P_1$  and  $P_2$ , such that all faces of  $P_1$  are removable in cw orientation with respect to  $l$  and all faces of  $P_2$  are removable in ccw orientation with respect to  $l$ , we can determine whether  $P_1$  and  $P_2$  are connected respectively in time  $O(n)$ .*

*Proof.* To test whether all the faces of  $P_1$  and  $P_2$  form connected components, consider two undirected graphs: graph  $G_1$  induced by the faces in  $P_1$  and graph  $G_2$  induced by the faces in  $P_2$ . Next, a depth-first search [15, Chapter 22] is performed on  $G_1$  and  $G_2$  respectively. The faces in  $P_1$  and  $P_2$  are connected, respectively, if and only if the depth-first forests of  $G_1$  and  $G_2$  consist of exactly one tree each. This implies that non-connected polyhedra are not castable with respect to any line. The running time for this algorithm is in the order of the sum of all the edges and vertices of the two undirected graphs, and hence  $O(n)$ .  $\square$

The final step is to determine whether  $P_1$  can be rotated by a small amount in cw orientation with respect to  $l$  without colliding with  $P_2$  and whether  $P_2$  can be rotated by a small amount in ccw orientation with respect to  $l$  without colliding with  $P_1$ . The definition of robust castability of  $P$  implies that  $P$  is robustly castable if and only if the rotations are possible.

**Lemma 4.** *Given a partitioning of  $\partial P$  into two connected components  $P_1$  and  $P_2$ , such that all faces of  $P_1$  are removable in cw orientation with respect to  $l$  and all faces of  $P_2$  are removable in ccw orientation with respect to  $l$ , we can test whether  $\exists \alpha > 0$  such that  $\forall q$  on  $P_1$*

$$cwar_c(l'(q), q, \alpha) \cap \text{int}(P_2) = \emptyset,$$

or equivalently  $\exists \alpha > 0$  such that  $\forall q$  on  $P_2$

$$ccwar_c(l'(q), q, \alpha) \cap \text{int}(P_1) = \emptyset$$

in time  $O(n)$ .

*Proof.* Each edge contained in the cycle  $C$  consisting of edges and vertices of  $P$  that separates  $P_1$  from  $P_2$  can be found in linear time by testing for each edge of  $P$  whether it is adjacent to both a face removable in cw orientation with respect to  $l$  and a face removable in ccw orientation with respect to  $l$ . The statement of Lemma 4 is true if and only if every point in  $C$  can rotate by a small amount in cw (respectively ccw) orientation with respect to  $l$  without penetrating  $P_2$  (respectively  $P_1$ ). As  $C$  consists of non intersecting simple polygonal cycles on  $P$ , it is sufficient to test all the vertices of  $C$ . Denote the intersection of the plane  $\pi$  perpendicular to  $l$  passing through a vertex  $p$  of  $C$  with  $l$  by  $l'(p)$ . Rotations of  $p$  with respect to  $l$  in 3-dimensional space correspond to rotations of  $p$  with respect to  $l'(p)$  in  $\pi$ . Previously, Bose et al. [8] examined rotational castability of simple polygons in two dimensions. An edge  $e$  of a polygon  $R$  with interior  $\text{int}(R)$  and boundary  $\partial R$  is *removable in cw orientation* with respect to a point  $r$  in the plane, if  $\exists \alpha > 0$  such that  $\forall p$  on  $e : cwar_c(r, p, \alpha) \cap \text{int}(R) = \emptyset$ . The edge  $e$  is *removable in ccw orientation* with respect to  $r$ , if  $\exists \alpha > 0$  such that  $\forall p$  on  $e : ccwar_c(r, p, \alpha) \cap \text{int}(R) = \emptyset$ . Bose et al. [8] proved that the valid orientation for cast removal with respect to  $r$  changes at a point  $p \in \partial R$  if and only if  $p$  either locally minimizes or maximizes the distance between  $\partial R$  and  $r$ . This result ensures that  $p$  can be rotated by a small amount in arbitrary orientation with respect to  $l'(p)$  if and only if  $p$  either locally minimizes or maximizes the distance between  $\partial P \cap \pi$  and  $l'(p)$  and  $p$  is not a reflex vertex in  $\pi$ . Hence,  $C$  can rotate by a small amount in cw (respectively ccw) orientation with respect to  $l$  without penetrating  $P_2$  (respectively  $P_1$ ), if every vertex  $p$  of  $C$  either locally minimizes or maximizes the distance between  $\partial P \cap \pi$  and  $l'(p)$  and is not a reflex vertex in  $\pi$ .

The aim is to test for every point  $p$  of  $C$  whether it either locally minimizes or maximizes the distance between  $\partial P \cap \pi$  and  $l'(p)$  and is not a reflex vertex in  $\pi$ . Note that  $p$  is incident to two edges in  $C$ , because  $C$  is a set of non-intersecting simple polygonal cycles on  $P$ . Hence, testing whether  $p$  is a local extremum requires intersecting  $\pi$  with the four faces adjacent to the two edges of  $C$  adjacent to  $p$ . We can test whether  $p$  minimizes or maximizes the distance to  $l'(p)$  by computing the distances from  $p$  and  $p$ 's two neighboring vertices in  $\pi$  to  $l'(p)$ . If  $p$  is closer to  $l'(p)$  than its two neighboring vertices, it locally minimizes the distance between  $\partial P \cap \pi$  and  $l'(p)$ . If  $p$  is further from  $l'(p)$  than its two neighboring vertices, it locally maximizes the distance between  $\partial P \cap \pi$  and  $l'(p)$ . Hence, testing whether one vertex  $p$  of  $C$  can rotate by a small amount in cw (respectively ccw) orientation with respect to  $l$  without penetrating  $\text{int}(P_2)$  (respectively  $\text{int}(P_1)$ ) takes constant time. As  $C$  has  $O(n)$  vertices, testing whether  $P_1$  and  $P_2$  collide takes  $O(n)$  time.  $\square$

This yields the following result:

**Theorem 1.** *Given a polyhedron  $P$  with combinatorial complexity  $n$  and a line of rotation  $l$  in 3-dimensional space, it is possible to determine whether  $P$  is robustly castable with respect to  $l$  in time  $O(n)$ .*

### 3.2 General rotational casting

Now, we turn our attention to general rotational casting. Recall that the only difference between a polyhedron being robustly rotationally castable and rotationally castable is the degenerate situation where faces or reflex edges of  $P$  are perpendicular to  $l$ . We now outline how to handle this situation which turns out to impose an additional  $\log n$  factor on the running time. Thus, determining whether a polyhedron  $P$  is castable with respect to a line  $l$  in space requires  $O(n \log n)$  time.

Consider the notation of Lemma 1. In the case of general castability of  $P$ , it is possible that a face  $f$  of  $P$  with inner normal  $\vec{n}$  is perpendicular to the given line  $l$ . The following Observation can easily be proven using the same techniques used in the proof of Lemma 1.

**Observation 1.** *Let  $f$  be a face of  $P$  with inner normal  $\vec{n}$  and let  $l$  be a line in direction  $\vec{l}$ . If  $l \cap S(f) = \emptyset$  and  $\vec{n} \times \vec{l} = \mathbf{0}$ ,  $f$  is removable in both cw and ccw orientation with respect to  $l$ .*

As in Section 3.1, it is possible to label all the faces of  $P$  that are not perpendicular to  $l$  as removable in cw, ccw, or no orientation in  $O(n)$  time. It remains to split the faces with  $l \cap S(f) = \emptyset$  and  $\vec{n} \times \vec{l} = \mathbf{0}$  and assign unique valid orientations to the newly created faces if possible. The face  $f$  can be partitioned into two (possibly empty) sets of faces  $F_1$  and  $F_2$ , such that all the faces adjacent to  $f$  removable in cw orientation with respect to  $l$  are adjacent to faces in  $F_1$  and all the faces adjacent to  $f$  removable in ccw orientation with respect to  $l$  are adjacent to faces in  $F_2$ . The definition of castability of a polyhedron implies that  $f$  is only removable with respect to  $l$  if there exists a partitioning of  $f$  into  $F_1$  and  $F_2$  with the following two properties:  $F_1$  does not intersect  $F_2$  when rotated by a small amount in cw orientation with respect to  $l$  and  $F_2$  does not intersect  $F_1$  when rotated by a small amount in ccw orientation with respect to  $l$ . Such a partitioning is called a *valid partitioning of  $f$* . The definition of castability implies that  $P$  is not castable with respect to  $l$  if no valid partitioning of  $f$  exists. Then, we say that  $f$  is not removable with respect to  $l$ .

If  $l$  passes through a hole of a face  $f$  perpendicular to  $l$ , there are two possible cases:  $l \cap \partial P \neq \emptyset$  and  $l \cap \partial P = \emptyset$ . If  $l \cap \partial P \neq \emptyset$ , at least one of the faces of  $P$  intersects  $l$  and is therefore not removable with respect to  $l$ . If  $l \cap \partial P = \emptyset$ , the boundary of the hole of  $P$  that contains  $l$  can not be removed without penetrating  $int(P)$ . This yields the following observation:

**Observation 2.** *If  $l$  passes through a hole of  $f$ ,  $P$  is not castable with respect to  $l$ .*

**Lemma 5.** *Let  $P$  be a polyhedron with combinatorial complexity  $n$  and let  $l$  be a line in direction  $\vec{l}$ . For all faces  $f$  of  $P$  with inner normal  $\vec{n}$ ,  $l \cap S(f) = \emptyset$ , and  $\vec{l}$  is parallel to  $\vec{n}$ , we can find a valid partitioning or report that  $f$  is not removable with respect to  $l$  in total time  $O(n \log n)$ .*

*Proof.* Consider a single face  $f$  with inner normal  $\vec{n}$ ,  $l \cap S(f) = \emptyset$ , and  $\vec{l}$  is parallel to  $\vec{n}$ . As  $f$  is located in a plane orthogonal to  $l$  and as no two adjacent faces of  $P$  are parallel, we know the valid orientation for cast removal with respect to  $l$  for every face adjacent to  $f$ . An edge of  $f$  is considered removable in cw (respectively ccw) orientation with respect to  $l$  if it is incident to a face adjacent to  $f$  removable in cw (respectively ccw) orientation with respect to  $l$ . Since the valid orientations with

respect to  $l$  of all the faces adjacent to  $f$  are known, every edge of  $f$  can be labeled as removable in cw or ccw orientation with respect to  $l$ , respectively, in constant time per edge. Note that no edge is labeled both cw and ccw because edges that were originally labeled this way have already been split as described above. The aim is to partition  $f$  into two or more components  $F_1$  and  $F_2$ , such that all the edges of  $f$  removable in cw orientation with respect to  $l$  are only incident to  $F_1$  and all the edges removable in ccw orientation with respect to  $l$  are only incident to  $F_2$ .

Denote the set of edges of  $f$  removable in cw orientation with respect to  $l$  by  $CW(f)$  and the set of edges of  $f$  removable in ccw orientation with respect to  $l$  by  $CCW(f)$ . Define  $cw\alpha_1(p) \geq 0$  as the maximal angle by which  $p$  can be rotated in cw orientation with respect to  $l$  without penetrating the exterior of  $f$ . Let  $cw\alpha_2 > 0$  be an arbitrarily small positive angle. Define  $cw\alpha(p) = \max(cw\alpha_1(p), cw\alpha_2)$  and note that  $cw\alpha(p) > 0$ . Let  $l'$  denote the point  $l \cap \pi$ , where  $\pi$  is the supporting plane of  $f$ .

**Claim 1.** *There exists a valid partitioning of  $f$  if and only if*

$$(cwarcl(l', p, cw\alpha(p)) \setminus \{p\}) \cap CCW(f) = \emptyset \forall p \in CW(f).$$

The proof of this claim is left until later.

Next, we present an algorithm to decide whether  $(cwarcl(l', p, cw\alpha(p)) \setminus \{p\}) \cap CCW(f) = \emptyset \forall p \in CW(f)$  for a given face  $f$  and analyze its running time. Each edge of  $f$  is labeled as removable in cw or ccw orientation with respect to  $l$  respectively. The aim is to test whether any point  $p$  on an edge removable in cw orientation with respect to  $l$  intersects an edge removable in ccw orientation with respect to  $l$  when rotated by  $cw\alpha(p)$  in cw orientation with respect to  $l$ . To conduct this test,  $f$  is represented in a fixed polar coordinate system, where  $l'$  is the origin and where the polar axis is not perpendicular to any edge of  $f$  and does not intersect  $f$ . If it is impossible to represent  $f$  that way, then  $P$  is not castable with respect to  $l$ , because the initial assumption ensures that  $l' \notin \text{int}(f)$  and Observation 2 shows that  $P$  is not castable with respect to  $l$  if  $l$  passes through a hole of  $f$ . In polar coordinates, every point is described by an angle  $\phi$  and the distance  $d$  from the origin. Transform the coordinate system into a cartesian coordinate system, such that the  $x$ -axis describes the angle  $\phi$  and the  $y$ -axis describes the distance  $d$ . Considering this transformed system between  $x = 0$  and  $x = 2\pi$  shows the transformed polar coordinate system. The polygon  $f$  is transformed into  $\tilde{f}$ . A sketch illustrating the transformation is shown in Figure 3. Since the positive polar axis does not intersect  $f$ ,  $f$  and  $\tilde{f}$  are topologically equivalent. The vertices of  $\tilde{f}$  are points and the edges of  $\tilde{f}$  are transformed line segments. The transformation of a line  $l$  is in essence a curve describing the distance of points on the line to the origin. The distance of points on  $l$  to the origin has exactly one minimum and consists of two strictly monotone pieces with monotone derivatives. Hence, there exists a pair  $(d_0, \phi_0)$  that denotes the minimum distance from the line to the origin and the distance increases symmetrically as  $\phi$  increases or decreases, respectively. This yields that edges of  $\tilde{f}$  have one global minimum and consist of at most two strictly monotone pieces. Furthermore, each edge  $e$  of  $f$  was split before (while splitting faces of type 4 in Lemma 1) at the perpendicular projection of  $l'$  on  $e$ . This corresponds to splitting each edge  $\tilde{e}$  of  $\tilde{f}$  at the global minimum. Hence, every edge of  $\tilde{f}$  is a strictly monotone curve with monotone derivative. Two edges of  $\tilde{f}$  only intersect in vertices of  $\tilde{f}$ . Each edge  $\tilde{e}$  has constant description size.

In the cartesian coordinate system, rotations in cw (respectively ccw) orientation with respect to  $l'$  correspond to translations in direction of the negative (respectively positive)  $x$ -axis. It is possible

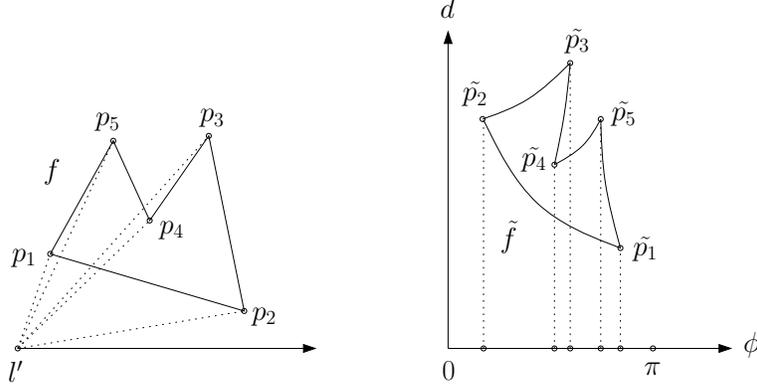


Figure 3: Sketch illustrates the transformation from  $f$  to  $\tilde{f}$ .

to determine whether  $f$  is removable with respect to  $l$  by enlarging a circle in the polar coordinate system. This corresponds to sweeping a line in  $y$ -direction in the cartesian plane. Let each edge  $\tilde{e}$  of  $\tilde{f}$  be labeled as removable in cw or ccw orientation with respect to  $l$  respectively and let  $\tilde{e}$  have a label indicating whether  $\text{int}(\tilde{f})$  is located to the left or to the right side of  $\tilde{e}$ . Let the sweep line status of the plane sweep algorithm contain the edges of  $\tilde{f}$  that are currently intersected by the sweep line and let the event queue contain all the vertices of  $f$ . At every event point of the plane sweep algorithm, an edge of  $\tilde{f}$  is either inserted to or removed from the sweep line status. If this insertion or removal, respectively, yields one of the following two situations, the algorithm reports that  $f$  is not removable with respect to  $l$ .

1. An edge  $\tilde{e}_1$  removable in cw orientation with respect to  $l$  is located immediately to the right of an edge  $\tilde{e}_2$  removable in ccw orientation with respect to  $l$  and  $\text{int}(\tilde{f})$  is located to the left of  $\tilde{e}_1$ .
2. An edge  $\tilde{e}_1$  removable in cw orientation with respect to  $l$  is located immediately to the right of an edge  $\tilde{e}_2$  removable in ccw orientation with respect to  $l$ ,  $\text{int}(\tilde{f})$  is located to the right of  $\tilde{e}_1$ , and  $\tilde{e}_1$  and  $\tilde{e}_2$  share a vertex.

If none of the above two situations occurs during the plane sweep, the algorithm reports that  $f$  is removable with respect to  $l$ .

The proof of correctness of this algorithm consists of two parts. The first part,  $f$  is not removable with respect to  $l$  if the algorithm finds one of the two above mentioned situations is easy to see. It remains to prove that  $f$  is removable with respect to  $l$  if none of the above mentioned situations occur. We prove this by contradiction and assume that  $f$  is not removable with respect to  $l$  although the two situations did not occur. Hence,  $\exists p \in CW(f)$ , such that  $(\text{cware}(l', p, \text{cw}\alpha(p)) \setminus \{p\}) \cap CCW(f) \neq \emptyset$ . Since the transformation from  $f$  to  $\tilde{f}$  maintains the distance from  $l'$ ,  $\exists p \in CW(\tilde{f})$ , such that  $(\text{cware}(l', p, \text{cw}\alpha(p)) \setminus \{p\}) \cap CCW(\tilde{f}) \neq \emptyset$ . Let the point  $q = (\text{cware}(l', p, \text{cw}\alpha(p)) \setminus \{p\}) \cap CCW(\tilde{f})$ . Since every edge of  $f$  is strictly monotone and has a monotone derivative,  $p$  and  $q$  are located on distinct edges  $\tilde{e}_1$  and  $\tilde{e}_2$  of  $\tilde{f}$  respectively. Two situations are possible. First,  $\text{cw}\alpha(p)$  is the cw angle that  $p$  is rotated by prior to leaving the interior of  $\tilde{f}$ . Assume without loss of generality that there is no edge between  $\tilde{e}_1$  and  $\tilde{e}_2$ . This assumption is valid, because if there is an edge between  $\tilde{e}_1$  and  $\tilde{e}_2$ , we can find an edge  $\tilde{e}_2'$  with the desired property. Hence,  $\tilde{e}_1$  is located immediately to the right of  $\tilde{e}_2$  and  $\text{int}(\tilde{f})$  is located to the left of  $\tilde{e}_1$ . This corresponds to the first situation

mentioned above and therefore contradicts the assumption. Second,  $cw\alpha(p)$  is an arbitrarily small angle, i.e.  $int(\tilde{f})$  is located to the right of the edge  $\tilde{e}_1$  and the distance between  $\tilde{e}_1$  and  $\tilde{e}_2$  is arbitrarily small. Hence,  $\tilde{e}_1$  and  $\tilde{e}_2$  share a vertex. This corresponds to the second situation mentioned above and therefore contradicts the assumption and proves the correctness of the algorithm.

For a face with  $s$  vertices, labeling all of the edges as removable in cw or ccw orientation with respect to  $l$  and transforming  $f$  to  $\tilde{f}$  takes time  $O(s)$ . The plane sweep algorithm takes time  $O(s \log s)$  [17, Chapter 2]. Since no two adjacent faces of  $P$  are parallel, every vertex of  $P$  is part of at most one face perpendicular to  $l$ . Hence, it takes  $O(n \log n)$  total time to test all of  $P$ 's faces with inner normal  $\vec{n}$ ,  $l \cap S(f) = \emptyset$ , and  $\vec{l}$  is parallel to  $\vec{n}$ .  $\square$

It remains to prove Claim 1:

*Proof.* The proof of this statement consists of two parts. First, we prove that  $f$  is not removable with respect to  $l$  if  $\exists p \in CW(f)$ , such that  $(cwarcl'(l', p, cw\alpha(p)) \setminus \{p\}) \cap CCW(f) \neq \emptyset$ . From the definition of  $cw\alpha(p)$ , we know that  $(cwarcl'(l', p, cw\alpha(p)) \setminus \{p\}) \cap \partial f$  is a single point  $q$ . The assumption ensures that  $q \in CCW(f)$ . If  $cwarcl'(l', p, cw\alpha(p)) \setminus (\{p\} \cup \{q\})$  is completely contained in the exterior of  $P$ ,  $cw\alpha(p)$  is an arbitrarily small angle. But this implies that  $p$  cannot be rotated by an arbitrarily small angle in cw orientation with respect to  $l$  without hitting a point of  $CCW(f)$ . The definition of castability implies that  $f$  is not removable. Otherwise,  $cwarcl'(l', p, cw\alpha(p)) \setminus (\{p\} \cup \{q\}) \in int(f)$ . Since  $p$  and  $q$  need to be removed in opposite orientations with respect to  $l$ , we find at least one point  $p^* \in cwarcl'(l', p, cw\alpha(p))$ , where the valid orientation with respect to  $l$  changes from cw to ccw when traversing  $cwarcl'(l', p, cw\alpha(p))$  starting at  $p$ . As the traversal from  $p$  to  $q$  is a rotation in cw orientation around  $l'$ ,  $p^*$  penetrates  $F_2$  (respectively  $F_1$ ) when rotated infinitesimally in cw (respectively ccw) orientation with respect to  $l$ . Therefore,  $f$  is not removable with respect to  $l$ .

The second part we prove is that there exists a valid partitioning of  $f$  if  $(cwarcl'(l', p, cw\alpha(p)) \setminus \{p\}) \cap CCW(f) = \emptyset \forall p \in CW(f)$ . We prove this explicitly by assigning a unique valid orientation with respect to  $l$  to every point  $q \in f$ , such that no cw point of  $f$  penetrates  $F_2$  when rotated by an arbitrarily small angle in cw orientation with respect to  $l$  and no ccw point of  $f$  penetrates  $F_1$  when rotated by an arbitrarily small angle in ccw orientation with respect to  $l$ . Consider an arbitrary point  $p \in CW(f)$ . Since  $(cwarcl'(l', p, cw\alpha(p)) \setminus \{p\}) \cap CCW(f) = \emptyset \forall p \in CW(f)$ , every point  $q$  on  $cwarcl'(l', p, cw\alpha(p))$  can rotate by an arbitrarily small angle in cw orientation with respect to  $l$  without hitting a point of  $CCW(f)$ . Hence, we label every point  $q \in int(f)$  with  $\exists p \in CW(f)$ , such that  $q \in cwarcl'(l', p, cw\alpha(p))$  as removable in cw orientation with respect to  $l$ . Every point  $q \in int(P)$  with  $q \notin cwarcl'(l', p, cw\alpha(p)) \forall p \in CW(f)$  can leave  $int(f)$  by a ccw rotation with respect to  $l$  without intersecting  $CW(f)$ . Every point  $q \in \partial P$  with  $q \notin cwarcl'(l', p, cw\alpha(p)) \forall p \in CW(f)$  can rotate by an arbitrarily small angle in ccw rotation with respect to  $l$  without intersecting  $CW(f)$ . Hence, we label every point  $q \in f$  with  $q \notin cwarcl'(l', p, cw\alpha(p)) \forall p \in CW(f)$  as removable in ccw orientation with respect to  $l$ . Since the two sets  $F_1 = \{q \mid \exists p \in CW(f), \text{ such that } q \in cwarcl'(l', p, cw\alpha(p))\}$  and  $F_2 = \{q \mid q \notin cwarcl'(l', p, cw\alpha(p)) \forall p \in CW(f)\}$  have the properties that  $F_1 \cap F_2 = \emptyset$  and  $f = F_1 \cup F_2$ , a valid partitioning of  $f$  was found. Note that the set of faces forming  $F_1$  and  $F_2$ , respectively, consists of faces bounded by edges of  $f$  and circular arcs with center point  $l'$ .  $\square$

This yields the conclusion that all the faces of a polyhedron  $P$  with combinatorial complexity  $n$  can be labeled as removable in cw, ccw, or no orientation with respect to a line of rotation  $l$  in space in  $O(n \log n)$  total time.

Since Lemmata 3 and 4 still hold in the general setting, we conclude with the following theorem:

**Theorem 2.** *Given a polyhedron  $P$  with combinatorial complexity  $n$  and a line of rotation  $l$  in 3-dimensional space, it is possible to determine whether  $P$  is castable with respect to  $l$  in time  $O(n \log n)$ .*

## 4 Determining all valid casting lines

In this section, we solve the problem of finding all lines  $l$  in 3-dimensional space, such that a given polyhedron is (robustly) castable with respect to  $l$ . Two aspects of the same problem are considered: reporting a representative for each class of combinatorially distinct (robust) valid casting lines for a given polyhedron and preprocessing  $P$  such that, for any given query line  $l$ , we can efficiently determine if  $P$  is (robustly) castable with respect to  $l$ . Two lines  $l_1$  and  $l_2$  are combinatorially distinct if the valid casting line  $l_1$  with respect to  $P$  can not continuously be moved to  $l_2$  without becoming an invalid casting line with respect to  $P$ .

### 4.1 Robust rotational casting

Recall that a polyhedron  $P$  is not robustly castable with respect to a line  $l$  with the property that some face or reflex edge of  $P$  is contained in a plane perpendicular to  $l$ . Hence, this section only considers lines  $l$ , such that no face or reflex edge of  $P$  is perpendicular to  $l$ .

As mentioned before, Bose et al. [8] consider rotational castability of polygons in two dimensions. They define a polygon to be *rotationally castable* with respect to a point  $r$  in the plane if the boundary of the polygon can be partitioned into exactly two connected chains, such that all the edges of one chain are removable in cw orientation with respect to  $r$  and all edges of the other chain are removable in ccw orientation with respect to  $r$ .

The 3-dimensional casting problem can be reduced to the 2-dimensional version considered by Bose et al. [8]. Consider an arbitrary plane  $\pi$  orthogonal to  $l$  and compute the set  $P' = \pi \cap P$ . The intersection consists of sets of polygons, since  $P$  is a polyhedron. Hence, the castability of  $P'$  with respect to the point  $l' = \pi \cap l \in \mathbb{R}^2$  can be analyzed using the methods proven by Bose et al. [8]. Since  $P'$  consists of a set of polygons,  $P'$  is castable with respect to  $l'$  if and only if all of the polygons in  $P'$  are castable with respect to  $l'$ . If a set of polygons forms a polygon with holes,  $P$  is not robustly castable with respect to  $l$ , because the polygon forming the hole can not be removed without penetrating  $int(P)$ .

Since  $P$  is a polyhedron, the boundary  $\partial P$  is an orientable surface. The proof of Lemma 3 states that  $P$  is not castable if  $\partial P$  is not connected. Hence, we assume in the following that  $\partial P$  is a connected orientable surface in  $\mathbb{R}^3$ .

**Definition 5.** The *genus  $g$  of a connected orientable surface  $S \in \mathbb{R}^3$*  is the maximum number of cuttings along closed simple curves without disconnecting  $S$ .

**Definition 6.** A *handle* is a subset of a manifold that is topologically equivalent to a ball.

The genus of a connected orientable surface  $S$  equals the number of handles of  $S$ . A *hole* in  $S$  exists if there exists a continuous loop  $L$  that cannot be homotopically deformed onto a point within the surface  $S$ . For each handle of  $S$ , there exists a hole in  $S$ , such that the corresponding loop  $L$  is partially located on the handle.

**Lemma 6.** *A polyhedron  $P$  in  $\mathbb{R}^3$  is robustly castable with respect to a line of rotation  $l$  if and only if every cross section  $P'$  of  $P$  with a plane  $\pi$  perpendicular to  $l$  is castable with respect to the point  $l' = \pi \cap l$ .*

*Proof.* The proof consists of two parts. First,  $P$  is not robustly castable with respect to  $l$  if any cross section  $P'$  is not castable with respect to  $l'$  because the rotation of  $\partial P$  around  $l$  includes the rotation of every possible  $\partial P'$  around  $l'$ .

Proving that  $P$  is robustly castable with respect to  $l$  if every cross section  $P'$  is castable with respect to  $l'$  requires showing that the points of  $\partial P'$  that locally minimize or maximize the distance from  $\partial P'$  to  $l'$  form connected chains that partition  $\partial P$  into exactly two connected sets. It follows then that all the faces of one component are removable in cw orientation with respect to  $l$  and all the faces of the other component are removable in ccw orientation with respect to  $l$ . Since the partitioning of  $\partial P$  is at near and far points that are not reflex vertices, Lemma 4 ensures that no collisions occur.

To examine the location of near and far points of  $\partial P'$ , the plane  $\pi$  perpendicular to  $l$  is swept over  $P$  in direction  $\vec{l}$ . When  $\pi$  is swept over  $P$ ,  $l'$  does not move and  $P'$  changes continuously. Hence, the near and far points of  $\partial P'$  with respect to  $l'$  move continuously along edges of  $\partial P$  and orthogonal projections of  $l$  on faces of  $\partial P$ . This implies that vertices of  $\partial P$  and the start and end points of the orthogonal projections of  $l$  on faces of  $\partial P$  are the only event points of the sweep algorithm. The output of the plane sweep is a set of circular lists representing cycles on  $\partial P$  formed by near and far points of  $\partial P'$  with respect to  $l'$ . The circular lists contain event points of the sweep algorithm ordered by their appearance on the cycle. During the plane sweep, a set of lists  $L$  that represent parts of circular lists is updated. When referring to event points of the plane sweep, only event points that were not swept yet are considered.

First, assume that only one event point occurs per cross section. The following events can occur at an event point  $v$  of the plane sweep:

1. In  $P'$ , a new polygon occurs as isolated vertex  $v$ . Hence,  $v$  is both a near and a far point with respect to  $l'$ . A new list that represents part of a circular list is created and added to the set  $L$ . It contains the event point adjacent to  $v$  with minimal distance from  $l$  (a near point with respect to  $l'$ ),  $v$ , and the event point adjacent to  $v$  with maximal distance from  $l$  (a far point with respect to  $l'$ ) in that order.
2. In  $P'$ , an existing polygon disappears as isolated vertex  $v$ . Hence,  $v$  is both a near and a far point with respect to  $l'$ . The event point  $v$  is already contained in one list of  $L$  as first point and in one (possibly the same) list of  $L$  as last point. If  $v$  is contained in two different lists, the lists are joined. Otherwise, the list of  $L$  containing  $v$  as both first and last point is stored as circular list.
3. In  $P'$ , an existing polygon splits into two polygons. The two polygons have the common vertex  $v$ , and therefore,  $v$  is both a near and a far point with respect to  $l'$ . A new list that represents part of a circular list is created and added to the set  $L$ . It contains the event point adjacent to  $v$  with minimal distance from  $l$ ,  $v$ , and the event point adjacent to  $v$  with maximal distance from  $l$  in that order.
4. In  $P'$ , two polygons merge into one. The two polygons have the common vertex  $v$ , and therefore,  $v$  is both a near and a far point with respect to  $l'$ . The event point  $v$  is already

contained in one list of  $L$  as first point and in one (possibly the same) list of  $L$  as last point. If  $v$  is contained in two different lists, the lists are joined. Otherwise, the list of  $L$  containing  $v$  as both first and last point is stored as circular list.

5. No topological changes occur in the cross section and the event point  $v$  is not contained in any list of  $L$ . Hence,  $v$  is neither a near nor a far point with respect to  $l'$  and no updates are required.
6. No topological changes occur in the cross section and the event point  $v$  is contained in a list of  $L$  as first or last point respectively. If  $v$  is the first point,  $v$  is a near point with respect to  $l'$ . The event point adjacent to  $v$  with minimal distance from  $l$  is added to the list of  $L$  containing  $v$  at the front. If  $v$  is the last point,  $v$  is a far point with respect to  $l'$ . The event point adjacent to  $v$  with maximal distance from  $l$  is added to the list of  $L$  containing  $v$  at the tail.

Note that although twelve topological changes can occur when a plane is swept over a polyhedron (see [10]), only the first four above are relevant as polygons with holes are not castable.

If a cross section contains more than one event point, the event points can simply be treated in sequential order, because no two event points are on one face or edge of  $P$ . As the set of circular lists  $L$  returned by the plane sweep represents chains of near and far points of  $\partial P'$  with respect to  $l'$ ,  $\partial P$  is partitioned along  $L$ . Hence, it remains to show that  $L$  partitions  $\partial P$  into exactly two connected sets. We prove this by induction on the genus  $g$  of  $\partial P$ .

Base case: There are no split or merge vertices. A simple polyhedron  $P$  (i.e.  $g = 0$ ) is robustly castable with respect to  $l$  if every cross section  $P'$  is castable with respect to  $l'$ . The plane sweep returns one simply connected cycle. We prove this by induction on the number of split and merge event points (event points of type 3 and 4).

- Base case: The first event point is of type 1 and creates one list that represents part of a circular list. As long as only event points of type 5 and 6 occur, no new lists are created. As there are no merge event points, it is impossible that further event points of type 1 occur. Hence, the only topological change that will occur when the last event point is swept is of type 2. This results in one circular list representing a simply connected cycle on  $\partial P$ . As  $\partial P$  has genus 0, any simply connected cycle partitions  $\partial P$  into exactly two connected sets.
- Induction step: Given that the plane sweep algorithm returns one circular list if  $k$  topological changes of type 3 or 4 occur during the plane sweep, we show that the plane sweep algorithm returns one circular list if  $k + 1$  topological changes of type 3 or 4 occur during the plane sweep. If the  $k + 1^{st}$  topological change is a split, a new list  $L_k$  is created. After the split, the two new polygons in  $P'$  disappear in event points of type 2. Note that they cannot merge again as  $P$  is simply connected. When the first of the two new polygons in  $P'$  disappears in an event point of type 2,  $L_k$  joins another list of  $L$ . The initial assumption implies that this list is eventually the only remaining list that will be returned as circular list. If the  $k + 1^{st}$  topological change is a merge, a new polygon occurs as an isolated vertex at an event point of type 1 creating a new list  $L_k$ . When this new polygon merges with another polygon,  $L_k$  joins another list of  $L$ . The initial assumption implies that this list is eventually the only remaining list that will be returned as circular list. Hence, the sweep algorithm always returns one simply connected circular list for a polyhedron of genus 0.

Induction step: Given that a polyhedron  $P_g$  with genus  $g$  is robustly castable with respect to  $l$  if every cross section  $P'_g$  is castable with respect to  $l'$ , we show that a polyhedron  $P_{g+1}$  with genus  $g + 1$  is robustly castable with respect to  $l$  if every cross section  $P'_{g+1}$  is castable with respect to  $l'$ . Adding a handle to the castable polyhedron  $P_g$  yields  $P_{g+1}$ . Hence, only event points incident to the hole adjacent to the newly created handle are relevant. The first event point that occurs is of type 3, i.e. a new list  $L_g$  is created. As long as only event points of type 5 and 6 occur, no new lists are created. It is impossible that an event point of type 1 or 2 occurs incident to a hole. If multiple event points of type 3 occur, the polygons in  $P'_{g+1}$  can therefore only merge again. Hence, one simply connected circular list representing a cycle on  $\partial P_{g+1}$  is returned when the sweep plane passed the hole. As all of the event points are incident to the hole adjacent to the newly created handle, the cycle does not intersect any of the existing cycles. Hence,  $P_{g+1}$  is partitioned into exactly two connected sets by the points in  $L$ , and is therefore robustly castable with respect to  $l$ . It remains to prove that by adding a handle to  $P_g$ , every possible polyhedron  $P_{g+1}$  of genus  $g + 1$  can be created. The Principal Theorem of surface topology [25, §39] states that two closed orientable surfaces are topologically equivalent if and only if they have the same genus. Hence, any polyhedron of genus  $g + 1$  is topologically equivalent to  $P_{g+1}$  and can therefore be created by adding a handle to  $P_g$ .  $\square$

For the two dimensional casting problem, Bose et al. [8] define black regions for edges and reflex vertices of a simple polygon as follows.

**Definition 7.** Let  $e$  be an edge of a simple polygon with vertices  $a$  and  $b$  and denote the inner normal of  $e$  by  $\vec{n}(e)$ . The open strip

$$\{\vec{p} + t\vec{n}(e) | p \in e \setminus \{a, b\}, t \geq 0\}$$

is called the *black region of  $e$* .

Let  $v$  be a reflex vertex of a simple polygon. Denote the two edges adjacent to  $v$  by  $e_1$  and  $e_2$  and denote their inner normals by  $\vec{n}(e_1)$  and  $\vec{n}(e_2)$  respectively. The near cone of  $v$  is defined as

$$\{\vec{v} + t_1\vec{n}(e_1) + t_2\vec{n}(e_2) | t_1, t_2 \geq 0, t_1 t_2 \neq 0\}$$

and the far cone of  $v$  is defined as

$$\{\vec{v} + t_1\vec{n}(e_1) + t_2\vec{n}(e_2) | t_1, t_2 < 0\}.$$

The *black region of  $v$*  is the union of the near cone and the far cone of  $v$ .

If the center of rotation  $r$  is located in the black region of an edge  $e$ ,  $e$  is not removable with respect to  $r$ . Similarly, if  $r$  is located in the black region of a reflex vertex  $v$ ,  $v$  can not be rotated by an infinitesimal angle around  $r$  without penetrating the interior of the polygon. Bose et al. [8] prove that a polygon is castable with respect to a point  $r$  in the plane if and only if  $r$  is not contained in the black region induced by edges and reflex vertices of the polygon.

In analogy to the 2-dimensional casting problem discussed by Bose et al. [8], it is possible to define black regions. For a polyhedron, black regions exist for faces, reflex edges, and reflex vertices.

**Definition 8.** For a face  $f$  of  $P$ , the *black region of  $f$*  is defined as  $S^+(f)$ .

Note that there is no valid casting line for  $f$  that properly intersects the black region of  $f$  (see Lemma 1). Figure 2 shows the black region of  $f$ , which is the prism that is swept when translating  $f$  along its inner normal vector.

**Definition 9.** Let  $e$  be a reflex edge of  $P$  and denote the inner normals of its two adjacent faces by  $\vec{n}_1$  and  $\vec{n}_2$ . Define the *near wedge* of  $e$  as

$$S^+(e) = \{\vec{p} + t_1\vec{n}_1 + t_2\vec{n}_2 : p \in cl(e) \text{ and } t_1, t_2 \geq 0\}$$

and the *far wedge* of  $e$  as

$$S^-(e) = \{\vec{p} + t_1\vec{n}_1 + t_2\vec{n}_2 : p \in cl(e) \text{ and } t_1, t_2 < 0\}.$$

The *black region* of a reflex edge  $e$  is defined as  $(S^+(e) \cup S^-(e)) \setminus e$ .

For every point  $r$  contained in the black region of a reflex edge  $e$ , there exists a point  $q \in e$  that locally minimizes or maximizes the distance from  $r$  to  $\partial P$ , respectively.

**Definition 10.** A vertex  $v$  of  $P$  is a *reflex vertex* of  $P$  if and only if all of  $v$ 's adjacent edges are reflex edges of  $P$ . Let  $v$  be a reflex vertex of  $P$  and denote the inner normals of its  $r$  adjacent faces by  $\vec{n}_1, \dots, \vec{n}_r$ . Define the *near cone* of  $v$  as

$$S^+(v) = \{\vec{v} + \sum_{i=1}^r t_i\vec{n}_i : t_i > 0 \forall i = 1, \dots, r\}$$

and the *far cone* of  $v$  as

$$S^-(v) = \{\vec{v} + \sum_{i=1}^r t_i\vec{n}_i : t_i < 0 \forall i = 1, \dots, r\}.$$

The *black region* of a reflex vertex  $v$  is defined as  $S^+(v) \cup S^-(v)$ .

For every point  $r$  contained in the black region of a reflex vertex  $v$ ,  $v$  locally minimizes or maximizes the distance from  $r$  to  $\partial P$ , respectively. Figure 4 illustrates the black region of a reflex edge and of a reflex vertex.

The black regions of faces, reflex edges, and reflex vertices of  $P$  are the *black regions induced* by  $P$ .

**Lemma 7.** A polyhedron  $P$  is robustly castable with respect to a line of rotation  $l$ , if and only if  $l$  does not intersect any of the black regions induced by  $P$ .

*Proof.* The proof consists of two parts. First, we prove that  $P$  is not robustly castable with respect to  $l$  if  $l$  intersects any of the black regions induced by  $P$ . If  $l$  intersects the black region induced by a face  $f$  of  $P$ ,  $l \cap S^+(f) \neq \emptyset$  and Lemma 1 states that  $P$  is not castable with respect to  $l$ . If  $l$  intersects the black region of a reflex edge  $e$  of  $P$ , there exists a point  $q$  on  $e$  that locally minimizes or maximizes the Euclidean distance from  $\partial P$  to  $l$ . If  $l$  is parallel to  $e$ , this is true for every point on  $e$ . Consider the intersection of  $P$  with the plane  $\pi$  perpendicular to  $l$  passing through  $q$ . In the cross-section,  $q$  is a reflex vertex that has extremal distance to  $\pi \cap l$ . Hence,  $\pi \cap l$  is contained in the two-dimensional black region of  $q$ . The results by Bose et al. [8] and Lemma 6 ensure that  $P$

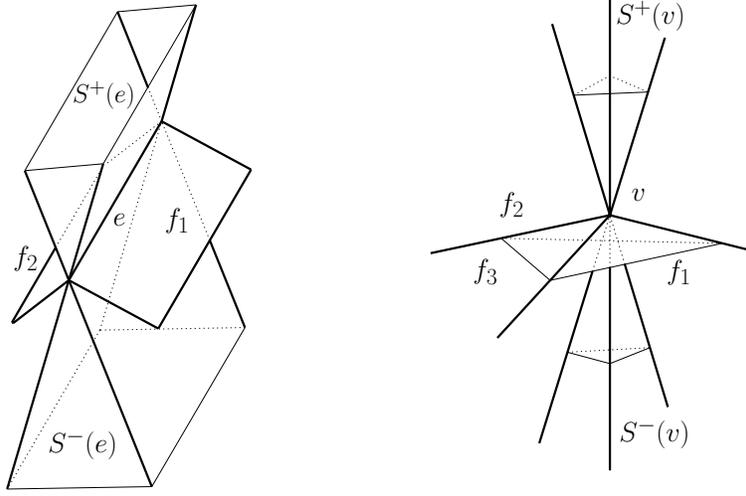


Figure 4: Left: The near and far wedge of  $e$ . Right: The near and far cone of  $v$ .

is not robustly castable with respect to  $l$ . If  $l$  intersects the black region induced by a reflex vertex  $v$  of  $P$ ,  $v$  locally minimizes or maximizes the Euclidean distance from  $\partial P$  to  $l$ . With an argument similar to the one we used for reflex edges, we can then show that  $P$  is not robustly castable with respect to  $l$ .

Second, we prove that  $P$  is robustly castable with respect to  $l$  if  $l$  does not intersect the union of black regions induced by  $P$ . We prove this by contradiction and assume that  $l$  does not intersect the union of black regions induced by  $P$  and that  $P$  is not robustly castable with respect to  $l$ . Since  $P$  is not robustly castable with respect to  $l$ , Lemma 6 ensures that there exists a cross section of  $P$  with a plane  $\pi$  perpendicular to  $l$  that is not castable with respect to  $\pi \cap l$ . Hence, there exists a cross section, such that  $\pi \cap l$  is contained in the 2-dimensional black region of the cross section. Three cases are possible:

1. The point  $l' = \pi \cap l$  is contained in the black region of an edge  $f' = \pi \cap f$ , where  $f$  is a face of  $P$  with normal  $\vec{n}$  not parallel to the direction  $\vec{l}$  of  $l$ . Denote the perpendicular projection of  $l'$  on  $\text{int}(f')$  by  $q$ . Since the normal vector of  $f'$  is the projection of  $\vec{n}$  onto the plane  $\pi$ , there exists a  $t \geq 0$ , such that

$$\vec{l}' = \vec{q} + \left( \vec{n} - \frac{\vec{n} \cdot \vec{l}'}{\vec{l}' \cdot \vec{l}'} \vec{l}' \right) t.$$

Consider the point  $l^*$  with

$$\vec{l}^* = \vec{l}' + t \frac{\vec{n} \cdot \vec{l}'}{\vec{l}' \cdot \vec{l}'} \vec{l}' = \vec{q} + t \vec{n}.$$

Clearly,  $l^* \in l$ , as  $l'$  reaches  $l^*$  by being moved in direction  $\vec{l}'$  only. Since  $q \in f$ ,  $l^*$  is also contained in the black region of  $f$ . This contradicts the initial assumption.

2. The point  $l' = \pi \cap l$  is contained in the black region of a reflex vertex  $e' = \pi \cap e$ , where  $e$  is a reflex edge of  $P$ . Since  $\pi \cap e$  is a vertex, none of the two faces adjacent to  $e$  is perpendicular to  $l$ . Denote the normal vectors of  $e$ 's two adjacent faces by  $\vec{n}_1$  and  $\vec{n}_2$  respectively. There

exist two constants  $t_1, t_2$  with  $t_1, t_2 \geq 0, t_1 t_2 \neq 0$  or  $t_1, t_2 < 0$ , such that

$$\vec{l}' = \vec{e}' + \left( \vec{n}_1 - \frac{\vec{n}_1 \cdot \vec{l}'}{\vec{l} \cdot \vec{l}'} \vec{l}' \right) t_1 + \left( \vec{n}_2 - \frac{\vec{n}_2 \cdot \vec{l}'}{\vec{l} \cdot \vec{l}'} \vec{l}' \right) t_2.$$

Consider the point  $l^*$  with

$$\vec{l}^* = \vec{l}' + t_1 \frac{\vec{n}_1 \cdot \vec{l}'}{\vec{l} \cdot \vec{l}'} \vec{l}' + t_2 \frac{\vec{n}_2 \cdot \vec{l}'}{\vec{l} \cdot \vec{l}'} \vec{l}' = \vec{e}' + t_1 \vec{n}_1 + t_2 \vec{n}_2.$$

Clearly,  $l^* \in l$ , as  $l'$  reaches  $l^*$  by being moved in direction  $\vec{l}'$  only. Since  $e' \in e$ ,  $l^*$  is also contained in the black region of  $e$ . This contradicts the initial assumption.

3. The point  $l' = \pi \cap l$  is contained in the black region of a reflex vertex  $v' = \pi \cap v$ , where  $v$  is a reflex vertex of  $P$ . Since  $v'$  is a reflex vertex,  $\pi$  intersects exactly two of  $v$ 's adjacent faces and neither of those faces is perpendicular to  $l$ . Hence, a proof similar to the previous proof shows that  $l$  intersects the black region of  $v$ . This contradicts the initial assumption.

Since every possible case yields a contradiction, the initial statement is proven to be true.  $\square$

Consider all the planes bounding black regions of faces, reflex edges, and reflex vertices of  $P$ . The black region of a face  $f$  of  $P$  is bounded by  $\deg(f)$  planes and can be expressed using  $\deg(f)$  lines perpendicular to  $f$  passing through vertices of  $f$ . The black region of a reflex edge  $e$  of  $P$  can be expressed using five lines: four lines perpendicular to the adjacent faces of  $e$  passing through the vertices of  $e$  and the supporting line of  $e$ . Finally, the black region of a reflex vertex  $v$  of  $P$  can be expressed using  $\deg(v)$  lines perpendicular to the faces adjacent to  $v$  passing through  $v$ . Hence, the arrangement  $A$  in  $\mathbb{R}^3$  containing for every face  $f$  of  $P$  the lines perpendicular to  $f$  passing through vertices of  $f$  and the supporting lines of the reflex edges of  $P$  describes all of  $P$ 's black regions. The arrangement  $A$  contains  $O(n)$  lines, since  $\sum_{f \in P} \deg(f) = 2E$ , where  $E$  is the number of edges of  $P$ . Two distinct lines  $g_1, g_2 \in A$  are in the same *equivalence class* of  $A$  if and only if it is possible to move  $g_1$  to  $g_2$  without crossing any of the lines forming  $A$ . If  $g_1$  does not cross any of the lines forming  $A$  during the transformation, it can not enter or leave any of the black regions induced by  $P$ . Assume that  $g_1$  and  $g_2$  are in the same equivalence class of  $A$ . Lemma 7 ensures that  $g_2$  is a valid casting line for  $P$  if and only if  $g_1$  is a valid casting line for  $P$ .

#### 4.1.1 Reporting all valid casting lines

In this section, we find and report a representative for each class of combinatorially distinct valid robust casting lines for a given polyhedron  $P$  with combinatorial complexity  $n$  in time  $O(n^4 \alpha(n))$ , where  $\alpha(n)$  is the inverse Ackermann function. The main idea used in this section is the fact that a line  $l$  in space is a valid robust casting line for  $P$  if and only if  $l$  does not intersect any black region induced by  $P$ , see Lemma 7.

**Theorem 3.** *Given a polyhedron  $P$  with combinatorial complexity  $n$ , it is possible to report all of the valid robust casting lines for  $P$  in  $\mathbb{R}^3$  in time  $O(n^4 \alpha(n))$ , where  $\alpha(n)$  is the inverse Ackermann function.*

*Proof.* To report all of the valid casting lines for  $P$  in  $\mathbb{R}^3$ , we construct the arrangement  $A$  of the lines defined as intersections of the planes bounding black regions induced by  $P$ . Every cell of  $A$  corresponds to exactly one equivalence class of  $A$ . Therefore, it is possible to label each cell  $c$  of  $A$  as an equivalence class of valid or invalid casting lines for  $P$ . Finally, a representing line for each equivalence class labeled as valid is reported.

The arrangement  $A$  of  $O(n)$  lines is constructed using a method by McKenna and O'Rourke [22]. They represent  $k$  lines in  $\mathbb{R}^3$  using four parameters per line and construct  $O(k^2)$  planar arrangements of hyperbolas in  $O(k^2\alpha(k))$  time each. They show that the arrangement has complexity  $\Theta(k^4)$  and can be constructed in time  $O(k^4\alpha(k))$ . The arrangement is represented as a graph  $G$ , where every line touching four of the  $k$  given lines or parallel to one of the  $k$  given lines and touching two of the remaining  $k - 1$  given lines is represented as a node of  $G$ . Using this method,  $A$  has complexity  $O(n^4)$  and can be constructed in time  $O(n^4\alpha(n))$ .

Once  $A$  and  $G$  have been constructed, every face of  $G$  needs to be labeled as valid or invalid. For this purpose, a boolean value is associated with every face  $f$ , reflex edge  $e$ , and reflex vertex  $v$  of  $P$  that indicates whether the current equivalence class of lines intersects the black region of  $f$ ,  $e$ , or  $v$  respectively. We start at an arbitrary face  $f_G$  of  $G$  and test for each face, reflex edge, and reflex vertex of  $P$  whether it causes the equivalence class of lines represented by  $f_G$  to be invalid. After testing, we set the boolean value of each face, reflex edge, and reflex vertex appropriately and compute the number  $b$  of faces, reflex edges, and reflex vertices that cause  $f_G$  to be invalid. Clearly,  $f_G$  is valid if and only if  $b = 0$ . This computation takes  $O(n)$  time as every face, reflex edge, and reflex vertex of  $P$  needs to be considered. Next,  $G$  is traversed in depth-first order. Each time, an edge  $e_G$  of  $G$  is crossed, we update the boolean values of the face, the reflex edge, and the reflex vertex of  $P$  that induce  $e_G$  and the counter  $b$ . This way, every face of  $G$  is labeled in constant time a piece. The edge  $e_G$  and its incident nodes represent valid casting lines of  $P$  if and only if one or more of  $e_G$ 's adjacent faces is labeled valid. Hence, all of the equivalence classes of  $A$  can be labeled in time  $O(n^4)$  and it is possible to report a representative for each class of combinatorially distinct valid casting lines for  $P$  in time  $O(n^4\alpha(n))$ .  $\square$

After computing  $A$  and  $G$  in  $O(n^4\alpha(n))$  preprocessing time, it is possible to perform line location in  $A$ , i.e. answer the question whether a given line  $l$  is a valid robust casting line for  $P$ . However, this query takes  $O(n)$  time [22]. Note that it is possible to report whether  $P$  is robustly castable with respect to a line  $l$  in time  $O(n)$  without preprocessing using Theorem 1. Therefore, a query time of  $O(\log n)$  is preferable. Hence, we use another approach to preprocess space for fast line location.

#### 4.1.2 Preprocessing space for fast line location

In this section, we preprocess the polyhedron  $P$  in  $O(n^{4+\epsilon})$  expected time in a way that allows to answer whether a query line is a valid casting line for  $P$  in time  $O(\log n)$ , where  $\epsilon$  is an arbitrarily small positive constant.

Using the same main idea as above, consider all the planes bounding black regions of faces, reflex edges, and reflex vertices of  $P$  and their  $O(n)$  intersecting lines. The arrangement of the lines is now represented in a way that allows fast queries.

For this purpose, each line is represented using Plücker coordinates, proposed in 1868 by Plücker [23]. In Plücker space, an oriented 3-dimensional line is described as a point called Plücker point in oriented projective 5-dimensional space using six coordinates. Dually, each line can also be

represented as a hyperplane called Plücker hyperplane in oriented projective 5-dimensional space. For two lines  $l$  and  $g$ , the Plücker point of  $l$  is located on the Plücker hyperplane of  $g$  if and only if  $l$  and  $g$  intersect and vice versa. That way, intersection tests of lines are linearized at the cost of operating in projective 5-dimensional space.

Consider a line  $l$  passing through two points  $p = (p_x, p_y, p_z, p_w)$  and  $q = (q_x, q_y, q_z, q_w)$  given in homogeneous space. The Plücker coordinates of the Plücker point corresponding to  $l$  are  $[l_0, l_1, l_2, l_3, l_4, l_5]$  and the Plücker coefficients of the Plücker hyperplane corresponding to  $l$  are  $[l_5, -l_4, l_3, l_2, -l_1, l_0]$  with  $l_0 = p_x q_y - p_y q_x$ ,  $l_1 = p_x q_z - p_z q_x$ ,  $l_2 = p_y q_z - p_z q_y$ ,  $l_3 = p_x q_w - p_w q_x$ ,  $l_4 = p_y q_w - p_w q_y$ , and  $l_5 = p_z q_w - p_w q_z$ .

Note that the vector  $\vec{l}_n = [l_0, -l_1, l_2]^T$  is the cross product  $\vec{p} \times \vec{q}$ , which is the normal vector of the plane passing through the line  $l$  and the origin. Furthermore, the vector  $\vec{l}_d = [l_3, l_4, l_5]^T$  is the difference vector  $\vec{p} - \vec{q}$ , which is the direction of the oriented line  $l$ .

The coordinates  $[l_0, l_1, l_2, l_3, l_4, l_5]$  specify a line in Plücker space if and only if

$$l_0 l_5 - l_1 l_4 + l_2 l_3 = 0. \quad (1)$$

This implies that not every set of 6-dimensional coordinates corresponds to a line in 3-dimensional space. In fact, in Plücker space, the set of all the lines corresponds to a 4-dimensional hypersurface.

Two lines  $l$  and  $g$  intersect if and only if

$$l_0 g_5 - l_1 g_4 + l_2 g_3 + l_3 g_2 - l_4 g_1 + l_5 g_0 = 0. \quad (2)$$

If the Plücker point of  $l$  is not located on the Plücker hyperplane of  $g$ , the sign of Equation (2) indicates the orientation of  $l$  in relation to  $g$ .

A survey on Plücker coordinates can be found in Stolfi's book [26, Chapter 19]. Chazelle et al. [12] applied Plücker coordinates in Computational Geometry to represent arrangements of lines in space. However, as they are only interested in small parts of the full arrangement, such as the envelope of the arrangement or single cells of the arrangement, they do not construct the full arrangement of lines in space.

**Theorem 4.** *A polyhedron  $P$  with combinatorial complexity  $n$  can be preprocessed in  $O(n^{4+\epsilon})$  expected time into a data structure of size  $O(n^{4+\epsilon})$  such that for any given line  $l$ , we can decide in  $O(\log n)$  time if  $P$  is robustly castable with respect to  $l$ , where  $\epsilon$  is an arbitrarily small positive constant.*

*Proof.* In a first step, all of the faces of  $P$  are triangulated. Since each face  $f$  of  $P$  is a possibly non-simple polygon, finding the constrained Delaunay triangulation and ignoring edges in the exterior of  $f$  yields a triangulation of  $f$ . Chew [13] and Wang and Schubert [28] independently proposed methods to find the constrained Delaunay triangulation of  $k$  vertices in time  $O(k \log k)$ . Since each edge has exactly two adjacent faces, it requires total time  $O(n \log n)$  to triangulate all of  $P$ 's faces. Since the triangulation adds  $O(k)$  new edges to each face, the combinatorial complexity of the triangulated polyhedron is  $O(n)$ .

The black regions of the triangular faces of  $P$  are defined in analogy to the black regions of faces of  $P$  before the triangulation. Let parallel lines intersect at infinity and augment the boundary of the black regions induced by  $P$  by vertices at infinity in every direction. The black region of a face of  $P$  is now bounded by four triangles. Furthermore, the black region of a reflex edge of  $P$  is bounded by eight triangles and the black region of a reflex vertex of  $P$  is bounded by  $s$  triangles, where  $s$  equals twice the degree of the vertex.

Observe that any line intersecting the black region induced by  $P$  must intersect the boundary of the black region induced by  $P$ , since none of the black regions induced by faces, reflex edges, and reflex vertices of  $P$  are unbounded along a line. As explained above, the boundary of the black region induced by  $P$  can be represented by  $O(n)$  triangles.

The question of whether a given query line intersects the black region induced by  $P$  is equivalent to the question of whether the query line intersects any of the bounding triangles of the black regions induced by  $P$ . Note that this problem is similar to the ray shooting problem arising in computer graphics, where a query ray is shot into a scene containing a set of triangles and the aim is to report the first triangle of the set hit by the query ray. However, our problem is easier than ray shooting, because we are only interested in knowing whether the query line intersects any of the triangles bounding the black regions induced by  $P$ .

For each of the  $O(n)$  edges of bounding triangles of the black region induced by  $P$ , the supporting lines are represented as hyperplanes in Plücker space. We then follow the approach proposed by de Berg et al. [16] to answer whether a given query line intersects any of the bounding triangles of the black region induced by  $P$ . Aronov et al. [4] showed that the complexity of the intersection of the arrangement of  $n$  Plücker hyperplanes in projective 5-dimensional space with the Plücker quadric expressed in Equation (1) is  $O(n^4 \log n)$ . Since we know that every query line is represented by a point inside the Plücker quadric, it suffices to consider the arrangement of hyperplanes representing supporting lines of triangles in the zone of the Plücker quadric. De Berg et al. use the randomized approach by Clarkson [14] to construct this 4-dimensional arrangement. Denote by  $H$  the set of hyperplanes representing supporting lines of triangles bounding the black regions induced by  $P$ . Let  $R \subset H$  be a sample of size  $O(r)$  with the property that any cell of the triangulated arrangement in projective 5-dimensional space is intersected by at most  $\frac{n}{r} \log r$  hyperplanes of  $R$ . Clarkson proved that such a sample can be obtained in  $O(nr^5)$  expected time by sampling repeatedly until the condition is met. A tree is built recursively as follows. The root of the tree is the arrangement  $A(R)$  induced by  $R$  in the zone of the Plücker quadric. Each of the  $O(r^4 \log r)$  children of the root corresponds to a cell of  $A(R)$ . If for a child of the root, there exists a triangle that is intersected by any line represented by a Plücker point in that cell, the child is labeled as invalid region and the recursion stops. Otherwise, the cell is recursively split and labeled. De Berg et al. [16] show that, by choosing  $r$  appropriately, the expected time and total space requirement to build this tree is  $O(n^{4+\epsilon})$ , where  $\epsilon$  is an arbitrarily small positive constant.

A given query line is represented as a point in Plücker space and point location is performed in the tree. When a leaf is reached, the label of the leaf reports whether the line intersects any of the triangles bounding the black region induced by  $P$ . Since a search in the tree requires  $O(\log n)$  time and the label can be read in constant time, the query time of the algorithm is  $O(\log n)$ .  $\square$

## 4.2 General rotational casting

In this section, we solve the problem of finding all lines  $l$  in 3-dimensional space, such that a given polyhedron  $P$  is castable with respect to  $l$ . Since the situation is identical to the situation in Section 4.1 if lines perpendicular to a face or an edge of  $P$  are neglected, this section focuses on lines perpendicular to a face or an edge of  $P$ .

As in Section 4.1, consider a plane  $\pi$  perpendicular to the line of rotation  $l$  and intersect  $\pi$  with  $P$ . The intersection  $P' = \pi \cap P$  consists of proper intersections, edges of  $P$  and faces of  $P$ . If a cross section  $P'$  does not contain any faces perpendicular to  $P$ ,  $P'$  can still be examined as before.

Recall that if  $P'$  contains a face of  $P$ , we need to test whether that face is removable with respect to  $l'$ . The test can be performed using Lemma 5. Hence, Lemma 6 can be changed to the following:

**Lemma 8.** *A polyhedron  $P$  in  $\mathbb{R}^3$  is castable with respect to a line of rotation  $l$  if and only if every cross section  $P'$  of  $P$  with a plane  $\pi$  perpendicular to  $l$  consists of castable and removable polygons with respect to  $l' = \pi \cap l$ .*

*Proof.* The main part of this Lemma was proven in the proof of Lemma 6. Hence, only faces perpendicular to  $l$  are discussed here. The proof of Lemma 5 shows that  $P$  is not castable if any face of  $P$  perpendicular to  $l$  is not removable with respect to  $l$ . Hence,  $P$  is not castable with respect to  $l$  if any of the cross sections  $P'$  contains a polygon that is not castable or not removable with respect to  $l'$ .

Next, we show that  $P$  is castable with respect to  $l$  if every cross section  $P'$  consists of castable and removable polygons with respect to  $l'$ . The same approach as in the proof of Lemma 6 is used and only faces perpendicular to  $l$  are discussed. If a cross section contains a face  $f$  perpendicular to  $l$ , we know that there exists a valid partitioning of  $f$ . If a new polygon occurs in  $P'$  as face  $f$ , treat the face like a vertex of type 1. If an existing polygon disappears in  $P'$  as face  $f$ , treat the face like a vertex of type 2. If an existing polygon splits into two or more polygons in  $P'$ ,  $f$  contains holes. Every arc contributing to a valid partitioning of  $f$  (see Lemma 5) that connects two holes of  $f$  is treated as a vertex of type 3. Arcs connecting to the outer face of  $f$  are treated as vertices of type 6. If two or more polygons merge into one in  $P'$ ,  $f$  contains holes. Every arc contributing to a valid partitioning of  $f$  that connects two holes of  $f$  is treated as a vertex of type 4. Arcs connecting to the outer face of  $f$  are treated as vertices of type 6. In each of the cases, the set of edges and arcs yielding a valid partitioning of  $f$  can be treated in sequential order. Hence, every case can be represented as a vertex of type 1 to 6. Therefore, the proof of Lemma 6 holds even when the line of rotation is perpendicular to a face or reflex edge of  $P$ .  $\square$

After introducing lines perpendicular to a face or edge of  $P$ , Lemma 7 does not hold any more. There are two reasons for this. First, there exist situations where  $P$  is castable with respect to  $l$  although  $l$  intersects black regions induced by  $P$ . As this cannot happen when faces and edges orthogonal to  $l$  are neglected (see Lemma 7), only those two situations need to be considered. If  $l$  intersects the black region of a face  $f$  perpendicular to  $l$ ,  $f$  is not removable with respect to  $l$  (see Lemma 1). Hence,  $P$  is not castable with respect to  $l$ . If  $l$  intersects the near wedge of a reflex edge  $e$  perpendicular to  $l$ , consider the cross-section of  $P$  with the plane  $\pi$  perpendicular to  $l$  containing  $e$ . The point  $l' = l \cap \pi$  is contained in the 2D black region of  $e$ . Hence, Lemma 6 implies that  $P$  is not castable with respect to  $l$ . However, if  $l$  only intersects the far wedge of a reflex edge  $e$  of  $P$  perpendicular to  $l$ ,  $P$  is castable with respect to  $l$ , because  $e$  can be split on the perpendicular projection of  $l$  on  $e$  and one part of  $e$  is removable in cw orientation with respect to  $l$  and the other part of  $e$  is removable in ccw orientation with respect to  $l$ . Second, there exist lines of rotation  $l$ , such that  $l$  does not intersect any of the black regions induced by  $P$  and  $P$  is not castable with respect to  $l$ . This can only occur, if  $P$  contains a face perpendicular to  $l$  that is not removable with respect to  $l$  (see Lemma 7). An example where a face perpendicular to  $l$  prevents  $P$  from being castable although  $l$  does not intersect any of the black regions induced by  $P$  is shown in Figure 5. This is an example where all of the faces can be removed, but the front face, which perpendicular to the line of rotation, cannot be split to allow its removal without intersecting one of the two protrusions.

Hence, Lemma 7 can be restated the following way:

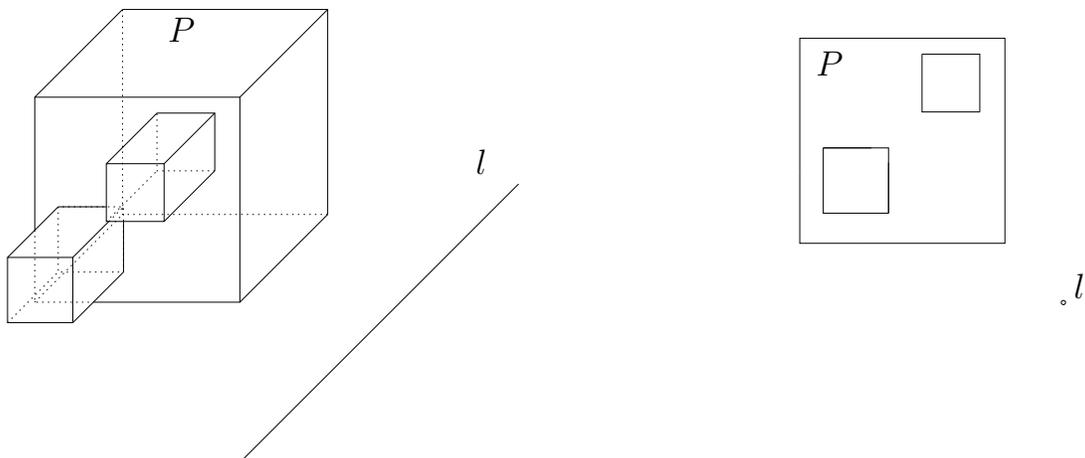


Figure 5: Example of polyhedron  $P$  that is not castable with respect to  $l$  although  $l$  does not intersect the black regions induced by  $P$ . Figure shows perspective view and front view.

**Lemma 9.** Let  $P$  be a polyhedron and let  $l$  be a line in space. The polyhedron  $P$  is castable with respect to  $l$  if  $l$  does not intersect any of the black regions induced by  $P$  and if every face of  $P$  perpendicular to  $l$  is removable with respect to  $l$ . If  $l$  intersects the black region induced by any face of  $P$ , the black region induced by any reflex vertex of  $P$ , or the near wedge of any reflex edge of  $P$ ,  $P$  is not castable with respect to  $l$ .

In analogy to Section 4.1, the aim is now to construct an arrangement  $A$  with the property that every equivalence class of  $A$  represents exactly one class of combinatorially distinct lines. To achieve this, the arrangement  $A$  used in Section 4.1 needs to be augmented to handle lines perpendicular to reflex edges and faces of  $P$ .

#### 4.2.1 Reporting all valid casting lines

In this section, we modify the results of Section 4.1 to find and report a representative for each class of combinatorially distinct valid casting lines for a given polyhedron  $P$  in time  $O(n^4 \log n)$ .

**Theorem 5.** Given a polyhedron  $P$  with combinatorial complexity  $n$ , it is possible to report all of the valid casting lines for  $P$  in  $\mathbb{R}^3$  in time  $O(n^4 \log n)$ .

*Proof.* In Section 4.1, the arrangement  $A$  represented by the graph  $G$  introduced by McKenna and O'Rourke [22] is constructed to represent all the classes of combinatorially distinct lines in space. The lines that are not perpendicular to a reflex edge or face of  $P$  can still be categorized and labeled using this technique. However, the arrangement needs to be extended to handle lines perpendicular to reflex edges and faces of  $P$ .

We will first discuss how to augment the arrangement to represent lines perpendicular to reflex edges of  $P$ . Recall that for any line perpendicular to a reflex edge of  $P$ , the far wedge of the reflex edge does not induce a black region. Hence, we need to insert all of the lines in space that are perpendicular to an edge of  $P$  into the arrangement  $A$ . The aim is to insert nodes into  $G$ , such that any time a line  $g$  that is not perpendicular to an edge  $e$  of  $P$  is moved continuously until  $g$  is perpendicular to  $e$ , a node of  $G$  is passed. McKenna and O'Rourke [22] prove that a line restricted

to intersect three skew lines has one degree of freedom. This degree of freedom can be used to move the line until it is perpendicular to  $e$ , unless every line touching the three skew lines is perpendicular to  $e$ . Consider a line touching three of the lines in the arrangement  $A$ . If every line touching the three skew lines is perpendicular to  $e$ , no node is inserted to  $G$ . The reason is that  $G$  already contains nodes that ensure that any time a line  $g$  that is not perpendicular to  $e$  is moved continuously until  $g$  is perpendicular to  $e$ , a node of  $G$  is passed. If there exists a line touching the three skew lines that is not perpendicular to  $e$ , the line touching the three skew lines perpendicular to  $e$  is inserted as node in  $G$ . For three given skew lines and an edge  $e$ , this line can be found in constant time, since every line touching three skew lines has one degree of freedom. To insert all of these nodes in  $G$ , the edges of  $G$  are traversed in depth-first order. An edge of  $G$  describes lines that are touching three skew lines. For each edge of  $G$ , find the  $O(n)$  nodes describing a line touching three skew lines and perpendicular to an edge  $e$  of  $P$ . This requires  $O(n)$  time. Then, sort the nodes according to their appearance on the edge in  $O(n \log n)$  time and insert them in order. It takes  $O(n \log n)$  time to insert the new nodes for one edge of  $G$ . Since  $G$  has  $O(n^3)$  edges, the total time to construct this arrangement is  $O(n^4 \log n)$ .

This arrangement needs to be augmented further to handle lines perpendicular to faces of  $P$  correctly. Lemma 9 implies that for any line  $l$  perpendicular to a face  $f$  of  $P$ ,  $P$  is only castable if  $f$  is removable with respect to  $l$ . For each face  $f$  of  $P$ , the arrangement  $A$  contains at least three lines perpendicular to  $f$  that bound the black region of  $f$ . Furthermore, the arrangement  $A$  contains every line touching four given lines or touching three given lines, where the third line is touched at infinity. Hence,  $G$  already contains all the nodes that represent lines perpendicular to faces of  $P$ .

It remains to label the faces of the new graph  $G$  we constructed. To label every face of  $G$ , traverse  $G$  in depth-first order. We store all the counters explained in Section 4.1 and an additional counter  $c$  of the number of far wedges of reflex edges perpendicular to lines in  $f_G$  that intersect the lines in  $f_G$ . This counter can be updated in constant time, if we store three boolean values with each reflex edge  $e$ : one value indicating whether the current line is in the near wedge of  $e$ , one value indicating whether the current line is in the far wedge of  $e$ , and one value indicating whether the current line is perpendicular to  $e$ . Furthermore, we store a boolean variable indicating whether the lines in  $f_G$  are perpendicular to a face  $f$  of  $P$ . If the lines represented by  $f_G$  are not perpendicular to any face of  $P$ , simply proceed as in Section 4.1. Initialize the counters once in  $O(n)$  time and update them in constant time per face. After subtracting  $c$  from the total number of black regions for  $f_G$ , we label  $f_G$  as valid if and only if the total number of black regions equals zero. If the lines represented by  $f_G$  are perpendicular to a face  $f$  of  $P$ , it is required to test whether  $f$  is removable with respect to the lines represented by  $f_G$ . For this purpose, we need to construct a planar arrangement for each face of  $P$ . Then, if the total number of black regions computed before equals zero, we need to look in the planar arrangement whether  $f$  is removable with respect to  $l$ .

For each set of faces  $F$  of  $P$  with  $f \in F$  has inner normal  $\pm \vec{n}$ , we construct a planar arrangement that subdivides a plane  $\pi$  with inner normal  $\vec{n}$  into points  $l' = l \cap \pi$  that are intersections of lines  $l$  in direction  $\vec{n}$ , such that each  $f \in F$  is removable with respect to  $l$  and points  $l' = l \cap \pi$  that are intersections of lines  $l$  in direction  $\vec{n}$ , such that  $\exists f \in F$  that is not removable with respect to  $l$ . Hence, every point  $l'$  in the plane  $\pi$  represents the line  $l$  perpendicular to  $\pi$  that intersects  $\pi$  in  $l'$ . To subdivide  $\pi$  into valid and invalid regions, several steps are required. First, the  $O(n)$  lines bounding black regions of  $P$  are projected to  $\pi$ . The planar arrangement has complexity  $O(n^2)$ . Every face of this arrangement can be labeled as black or non-black region, respectively, in  $O(n^2)$  time by walking through the arrangement and maintaining a counter of all intersected black

regions using a similar technique as explained in Theorem 3. For every face of the arrangement labeled non-black, we need to test for each  $f \in F$  whether  $f$  is removable with respect to lines represented by that face. Let  $p_1$  and  $p_2$  denote two arbitrary points inside the same face labeled as non-black of the arrangement. Since  $p_1$  and  $p_2$  are in the same face of the arrangement, all the faces of  $P$  have the same valid orientation with respect to the two lines represented by  $p_1$  and  $p_2$ , that is, the faces are either removable in cw orientation with respect to both lines, removable in ccw orientation with respect to both lines, or need to be split with respect to both lines. Hence, if the algorithm described in Lemma 5 is used to determine whether  $f$  is removable with respect to the line represented by  $p_1$  and  $p_2$ , respectively, all the edges of  $f$  are labeled in the same way in the first step. Hence, the two transformed polar planes with origins  $p_1$  and  $p_2$ , respectively, have the same structure. Although the actual distances to  $p_1$  and  $p_2$  are different, the same sequence of monotone curves with monotone derivatives occurs in the transformed plane. Furthermore, if an edge is split, the actual split vertex is different, but the situation in the transformed plane remains the same. Therefore,  $f$  is removable with respect to  $p_2$  if and only if  $f$  is removable with respect to  $p_1$ . This implies that for each face of the arrangement labeled as non-black, we need to test one representative using the algorithm of Lemma 5 in  $O(r \log r)$  time, where  $r$  is the number of edges of  $f$ . Let  $s$  denote the cardinality of  $F$ . In  $O(rs \log r)$  time, it is tested whether each  $f \in F$  is removable with respect to lines represented by one face of the arrangement labeled as non-black. Hence, it requires  $O(n^2 rs \log r)$  time to construct and label the arrangement for the faces  $F$ . This step needs to be done for each of the  $O(n)$  sets of faces with codirectional inner normals. In the worst case, both  $s$  and  $r$  are in the order of  $n$ . This seems to imply that the time complexity to construct and label the arrangements for all faces of  $P$  is  $O(n^5 \log n)$ . However, each face of  $P$  is represented in exactly one planar arrangement and each edge of  $P$  is shared by exactly two faces. Therefore, the sum of all the cardinalities and the sum of all the edges are both in the order of  $n$ . This implies that the total time complexity to construct and label the arrangements for all faces of  $P$  is  $O(n^4 \log n)$ .

This implies that in  $O(n^4 \log n)$  time, we can find and report all of the combinatorially distinct valid casting lines for  $P$ .  $\square$

After computing  $A$  and  $G$  in  $O(n^4 \log n)$  preprocessing time, it is possible to perform line location in  $A$ , i.e. answer the question whether a given line  $l$  is a valid casting line for  $P$ . A query in  $A$  requires  $O(n)$  time [22]. If the face of  $A$  contains the information that  $l$  is perpendicular to a set  $F$  of faces of  $P$ , it is necessary to query the arrangement induced by  $F$  to determine whether each  $f \in F$  is removable with respect to  $l$ . It is possible to determine  $l' = \pi \cap l$  in constant time and to determine the face of the arrangement induced by  $F$  that contains  $l'$  in  $O(\log n)$  time [21]. Hence, it takes  $O(n + \log n) = O(n)$  time to report whether  $P$  is castable with respect to  $l$ . Note that this implies that we can report faster whether  $P$  is castable with respect to  $l$  than without preprocessing (see Theorem 2). However, a query time of  $O(\log n)$  is preferable. Hence, we use another approach to preprocess space for fast line location.

#### 4.2.2 Preprocessing space for fast line location

In this section, we preprocess  $P$  in expected time  $O(n^{4+\epsilon})$  in a way that allows to answer whether a query line is a valid casting line for  $P$  in time  $O(\log n)$ , where  $\epsilon$  is an arbitrarily small positive constant.

As before, consider all of the planes bounding the black regions induced by  $P$  and their  $O(n)$  intersecting lines. Again, we use ray shooting in Plücker space to determine whether the boundary of any black region is intersected by the query line. However, the data structures become more complex, since lines perpendicular to faces and reflex edges of  $P$  need to be taken into consideration.

**Theorem 6.** *A polyhedron  $P$  with combinatorial complexity  $n$  can be preprocessed in  $O(n^{4+\epsilon})$  expected time into a data structure of size  $O(n^{4+\epsilon})$  such that, for any given line  $l$ , we can decide in  $O(\log n)$  time if  $P$  is castable with respect to  $l$ , where  $\epsilon$  is an arbitrarily small positive constant.*

*Proof.* We employ three data structures to answer whether a query line  $l$  is a valid casting line for  $P$ . The first data structure represents the black regions induced by faces of  $P$ , the black regions induced by reflex vertices of  $P$ , and the near wedges induced by reflex edges of  $P$ . By Lemma 9, any query line intersecting the boundary of the black regions represented in the first data structure is an invalid casting line for  $P$ . Hence, we can use the same approach as in the proof of Lemma 4 to build a data structure of size  $O(n^{4+\epsilon})$  in expected time  $O(n^{4+\epsilon})$  that answers whether  $l$  intersects any of the boundaries of black regions represented in the first data structure in  $O(\log n)$  time. If the answer to the query reports that  $l$  intersects any of the black regions represented by the first data structure,  $l$  is an invalid casting line and we are done. Otherwise, the second data structure is queried.

The second data structure considers the far wedges of the reflex edges of  $P$ . For all of the far wedges of reflex edges of  $P$  that are not perpendicular to  $l$ , de Berg et al.'s data structure [16] can be used as in the proof of Lemma 4 to answer whether  $l$  intersects the boundary of those far wedges. Recall that de Berg et al.'s data structure has size  $O(n^{4+\epsilon})$  and answers queries in  $O(\log n)$  time.

Each reflex edge of  $P$  is associated with four triangles bounding the far wedge of that reflex edge. The direction of a reflex edge of  $P$  can be represented as a point on the upper unit hemisphere of directions, because the orientation of the edge can be neglected. Using central projection from the upper unit hemisphere onto the plane  $\pi : z = 1$  with the origin as center of projection yields a representation of the directions of the reflex edges of  $P$  in 2-dimensional Euclidean space. Note that each reflex edge is represented as a point in  $\pi$ , which may lead to more than one reflex edge mapping to one point in  $\pi$ . This representation has the property that directions that lie on a great arc of the upper unit hemisphere of directions correspond to points lying on a line in  $\pi$ . The directions in the upper unit hemisphere perpendicular to  $l$  can be represented as a great arc of the upper unit hemisphere of directions or as a line  $l'$  in  $\pi$ . Hence, all of the reflex edges of  $P$  that are not perpendicular to  $l$  can be found using two half-space queries in  $\pi$  with the two open half-spaces bounded by  $l'$ . The triangles bounding the far wedges of the reflex edges of  $P$  that are not perpendicular to  $l$  can be used in de Berg et al.'s data structure [16] to answer whether  $l$  intersects the boundary of those far wedges.

For the second data structure, note that we have a set of  $O(n)$  points in 2-dimensional Euclidean space associated with four triangles each. We can therefore use a two-level data structure based on a cutting tree [17, Chapter 16]. A  $\frac{1}{r}$ -cutting in the plane is a collection of pairwise disjoint planar simplices that cover the entire plane with the property that each simplex intersects at most  $\frac{n}{r}$  of the  $n$  given lines. A cutting tree on  $n$  lines in the plane is recursively defined as follows: if  $n = 1$ , then the tree consists of a single node storing the line. Otherwise, a  $\frac{1}{r}$ -cutting is found and for each simplex of the  $\frac{1}{r}$ -cutting, we store a child of the current node. That child contains the simplex that

created the child, a list of all of the lines located completely above the simplex, and a list of all of the lines located completely below the simplex. Furthermore, each child is recursively defined as the root of a cutting tree on the lines crossing the simplex associated with the child [17, Chapter 16]. Chazelle [11] proved that a  $\frac{1}{r}$ -cutting in the plane of size  $O(r^2)$  can be found in  $O(nr)$  time.

The cutting tree used in the two-level data structure stores the set  $H$  of lines dual to the 2-dimensional points representing the reflex edges of  $P$ . Every node of the cutting tree stores a set  $H' \subset H$  of lines corresponding to a set of triangles bounding the far wedges of the reflex edges associated with the lines  $H'$ . These triangles are stored in de Berg et al.'s data structure [16]. Van Kreveld's results [27, Theorem 5.8] imply that, for an appropriate choice of  $r$ , the total size of the cutting tree storing de Berg et al.'s data structure in each node is  $O(n^\epsilon n^{4+\epsilon})$  for any  $\epsilon > 0$ . Replacing  $\epsilon$  by  $\frac{\epsilon}{2}$  yields the space requirement  $O(n^{\frac{\epsilon}{2}} n^{4+\frac{\epsilon}{2}}) = O(n^{4+\epsilon})$ .

To analyze the time requirement to build the second data structure, we use that a  $\frac{1}{r}$ -cutting in the plane of size  $O(r^2)$  can be found in  $O(nr)$  time [11]. We use the fact that by a similar analysis as the one used by van Kreveld [27, Theorem 5.8], this yields the following recurrence for the expected construction time  $P(n)$  of the data structure:  $P(n) \leq c_1 r^2 P(\frac{n}{r}) + c_2 r^{c_3} g(n) + c_4 nr$ , where  $c_1, \dots, c_4$  are constants and where  $g(n) = n^{4+\epsilon}$  is the expected preprocessing time required to build de Berg et al.'s data structure [16]. By choosing  $r$  to be a sufficiently large constant, this recurrence solves to  $O(n^\epsilon(n^2 + g(n)) + n^2 \log n)$  for any  $\epsilon > 0$ . If we replace  $\epsilon$  by  $\frac{\epsilon}{2}$ , the data structure therefore requires  $O(n^{4+\epsilon})$  expected construction time.

For a query line  $l$ , we perform two queries in that data structure. The line  $l'$  in  $\pi$  that represents all the directions perpendicular to  $l$  is found. We query the data structure with both open half-spaces bounded by  $l'$ . This is done by querying the data structure twice with the dual point of  $l'$  and by reporting all of the nodes of the cutting tree representing only lines of  $H$  that lie on the correct side of the dual query point, respectively. Then,  $l$  is represented as Plücker point and a query is performed in each of the secondary structures corresponding to the reported nodes. Van Kreveld's results [27, Theorem 5.8] imply that this data structure supports queries in  $O(\log n)$  time.

If the answer to the query reports that  $l$  intersects any of the black regions represented by the second data structure,  $l$  is an invalid casting line and we are done. Otherwise, the third data structure is queried.

The third data structure tests whether each face of  $P$  perpendicular to the query line  $l$  is removable with respect to  $l$ . As in the proof of Theorem 5, for each set of faces  $F$  of  $P$  with  $f \in F$  has inner normal  $\pm \vec{n}$ , we construct a planar arrangement that subdivides a plane with inner normal  $\vec{n}$  into points  $l' = l \cap \pi$  that are intersections of lines  $l$  in direction  $\vec{n}$ , such that each  $f \in F$  is removable with respect to  $l$  and points  $l' = l \cap \pi$  that are intersections of lines  $l$  in direction  $\vec{n}$ , such that  $\exists f \in F$  that is not removable with respect to  $l$ . We know from the proof of Theorem 5 that in  $O(n^4 \log n)$  time, we can build the planar arrangements with total space requirement  $O(n^3)$  for all the sets  $F$ . For each set  $F$ , represent the directions  $\pm \vec{n}$  in a dictionary with the property that the set  $F$  defined by a given direction  $\vec{d}$  can be found in  $O(\log n)$  time, if it exists.

Querying this data structure is slightly more complicated than in the proof of Theorem 5. For a query line  $l$ , query the dictionary with the direction of the query line to find the set of faces  $F$  of  $P$  perpendicular to  $l$ . If there are no faces perpendicular to  $l$ ,  $l$  is a valid casting line for  $P$ . Otherwise, compute the intersection  $l'$  of  $l$  with the planar arrangement associated with  $F$  and perform point location. This requires  $O(\log n)$  time [21]. Retrieving the label of the face of the planar arrangement containing  $l'$  answers whether all of the faces of  $F$  are removable with respect to  $l$ . Hence, querying the third data structure takes time  $O(\log n)$ .

Hence, the total space requirement and expected preprocessing time for the three data structures is  $O(n^{4+\epsilon})$ . Queries can be answered in  $O(\log n)$  total time.  $\square$

### 4.3 A Class of Polyhedra with $\Omega(n^4)$ Combinatorially Distinct Valid Casting Lines

In this section, we examine the complexity of all the combinatorially distinct valid (robust) casting lines for  $P$ . We show that there exists a class of polyhedra where the number of combinatorially distinct valid (robust) casting lines is  $\Omega(n^4)$ . This implies that the algorithms summarized in Theorems 3 and 5 are almost worst case optimal. We now outline the construction of the lower bound.

First, we construct a cylinder  $C$  in  $z$ -direction. The cross section of  $C$  with a plane  $z = z_0$  perpendicular to the  $z$ -axis is a simple polygon  $C'$  consisting of  $r = 3s - 1$  vertices located on two different polygonal chains. Let  $s$  vertices of  $C'$  be evenly distributed on the upper half of the unit circle. The coordinates of those vertices are

$$(\cos((i-1)\phi_1), \sin((i-1)\phi_1), z_0), \quad i = 1, \dots, s,$$

where  $\phi_1 = \frac{\pi}{s-1}$ . Hence, the vertices form a convex polygonal chain  $c_1$ . The black regions of the faces of  $C$  inducing the cross section  $c_1$  can be projected onto the plane  $z = z_0$ . The projected black regions induce valid regions that are cones whose apexes  $a$  are on the unit circle and whose opening angles are  $\frac{\phi_1}{2}$ , see Figure 6.

The second polygonal chain  $c_2$  of  $C'$  consists of  $2s - 1$  vertices. Let  $s$  vertices of  $c_2$  be evenly distributed on the arc of the circle with center  $(-\frac{1}{2}, 0, z_0)$  and radius 1 starting at  $\frac{3\pi}{2}$  and ending at  $\frac{25\pi}{16}$ . The coordinates of those vertices are

$$\left(-\frac{1}{2} + \cos\left(\frac{3\pi}{2} + (i-1)\phi_2\right), \sin\left(\frac{3\pi}{2} + (i-1)\phi_2\right), z_0\right), \quad i = 1, \dots, s,$$

where  $\phi_2 = \frac{\pi}{16(s-1)}$ . Denote the vertices by  $v_1, \dots, v_s$  and note that  $v_i$  is not located in the interior of the unit disk for  $i = 1, \dots, s$ . Define the vertices  $v_0, v_{s+1}$  as

$$\left(-\frac{1}{2} + \cos\left(\frac{3\pi}{2} - \phi_2\right), \sin\left(\frac{3\pi}{2} - \phi_2\right), z_0\right), \quad \left(-\frac{1}{2} + \cos\left(\frac{3\pi}{2} + s\phi_2\right), \sin\left(\frac{3\pi}{2} + s\phi_2\right), z_0\right),$$

respectively. Let  $s - 1$  vertices of  $c_2$  be defined as the intersections of the line passing through  $v_{i-1}$  and  $v_i$  with the line passing through  $v_{i+1}$  and  $v_{i+2}$ , where  $i = 1, \dots, s - 1$ . These vertices are located on a circle. The polygonal chain  $c_2$  consists of  $s - 2$  reflex,  $s - 1$  convex, and 2 boundary vertices. Note that  $c_2$  consists of sides of isosceles triangles, i.e. all the edges have the same length, see Figure 7. The black regions of the faces of  $C$  inducing the cross section  $c_2$  can be projected onto the plane  $z = z_0$ . The projected black regions induce valid regions bounded by part of  $c_2$  and two parallel half lines.

In  $C'$ , the two polygonal chains  $c_1$  and  $c_2$  can now be connected by two edges. This does not introduce further reflex vertices to  $C'$ , but only two black regions of the two faces of  $C$  inducing the new edges of  $C'$ . Those black regions have no influence on further considerations. Each of the black regions induced by the reflex edges with cross section  $c_2$  induces a bounded valid region in the plane  $z = z_0$  when intersecting the valid region induced by the faces with cross section  $c_1$  starting at  $\frac{7\pi}{16}$  and ending at  $\frac{\pi}{2}$ . Hence, there are at least  $(s - 2) \lfloor \frac{s}{8} \rfloor$  bounded valid regions. Since  $C'$  consist of  $r = 3s - 1$  vertices, there are  $\frac{r-5}{3} \lfloor \frac{r+1}{24} \rfloor = \Omega(r^2)$  bounded valid regions in the plane  $z = z_0$ .

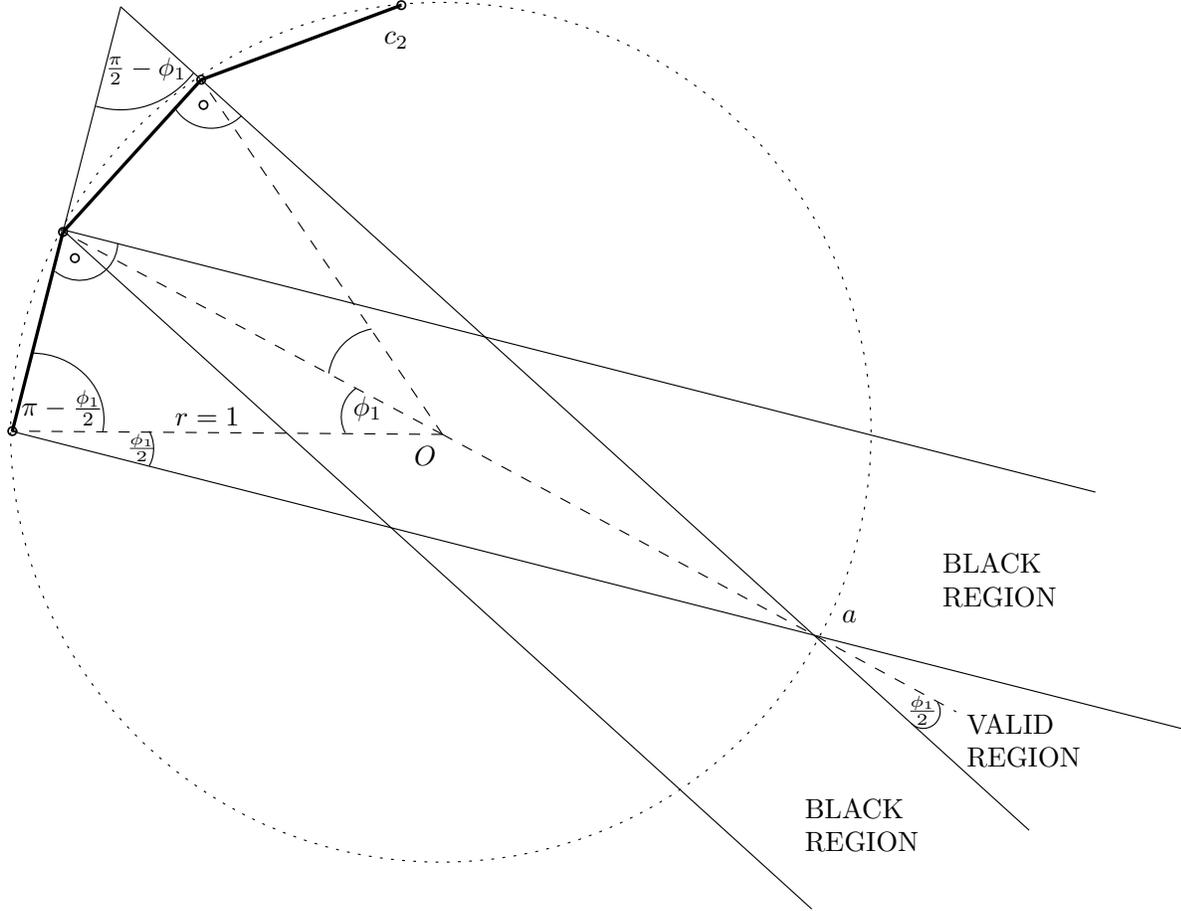


Figure 6: Approximation of half circle

An example of a cross section  $C'$  with  $s = 10$  is shown in Figure 8. Note that the described cross section  $C'$  gives a quadratic lower bound in two dimensions. This lower bound was first published by Bose et al. [8].

The polyhedron  $P$  consists of  $n = 4r + 8$  vertices. Two of the cylinders explained above are joined by a small cylinder whose cross section is a square. The first cylinder  $C_1$  with cross section  $C'$  is located between the two planes  $z = 0$  and  $z = 1$ . The second cylinder  $C_2$  is located between the two planes  $z = 2$  and  $z = 3$ . The two cylinders  $C_1$  and  $C_2$  are joined by a box with the corners  $(-\frac{3}{4}, 0, 1)$ ,  $(-\frac{9}{16}, 0, 1)$ ,  $(-\frac{9}{16}, \frac{3}{16}, 1)$ ,  $(-\frac{3}{4}, \frac{3}{16}, 1)$ ,  $(-\frac{3}{4}, 0, 2)$ ,  $(-\frac{9}{16}, 0, 2)$ ,  $(-\frac{9}{16}, \frac{3}{16}, 2)$ , and  $(-\frac{3}{4}, \frac{3}{16}, 2)$ . For an illustration of  $P$ , refer to Figure 9.

The black regions induced by the faces and reflex edges of the box needed to join the two cylinders  $C_1$  and  $C_2$  are completely located in  $x = d_1, d_1 \in [-\frac{3}{4}, -\frac{9}{16}]$  and  $y = d_2, d_2 \in [0, \frac{3}{16}]$ . Hence, any line passing through one of the  $\Omega(r^2)$  bounded valid regions induced by  $C_1$  and passing through one of the  $\Omega(r^2)$  bounded valid regions induced by  $C_2$  is a valid casting line for  $P$ . Hence, there exist  $\Omega(r^4)$  combinatorially distinct valid casting lines for  $P$ . Since none of the bounded valid regions induced by  $C_1$  and  $C_2$  restricts a line passing through the valid region to be perpendicular to a face of  $P$ , there are also  $\Omega(r^4)$  combinatorially distinct valid robust casting lines for  $P$ . Since  $r = \frac{n-8}{4}$ , there exist  $\Omega(n^4)$  combinatorially distinct valid (robust) casting lines for  $P$ .



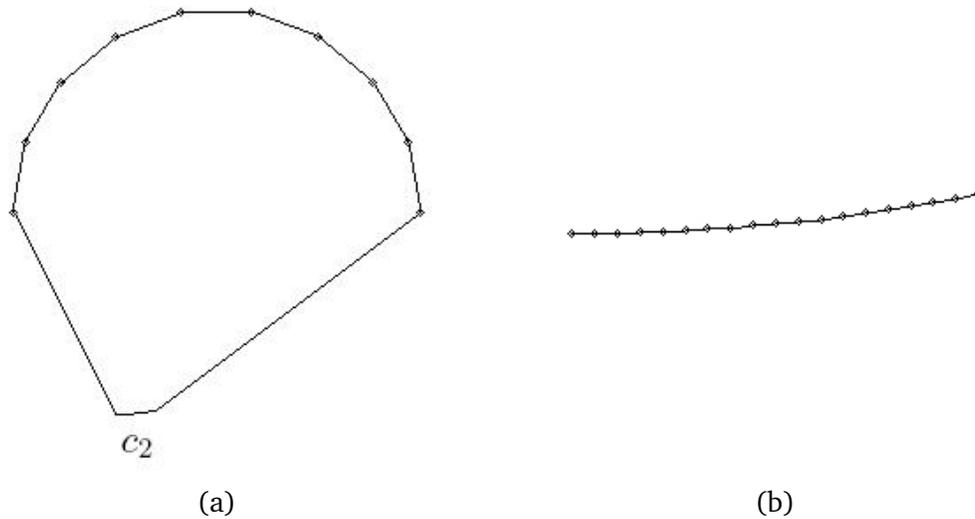


Figure 8: Example with  $s = 10$ . (a) shows the cross section, (b) shows an enlargement of the polygonal chain  $c_2$ .

- The control of the physical casting machinery is imperfect. This yields to surface defects if the cast of the object slides along the boundary of the object. The algorithms we presented should therefore be extended to determine whether an object is castable in a way that the cast does not slide along the boundary of the object.
- Since many objects do not have a piecewise linear boundary, the algorithms should be extended to handle more general object boundaries.

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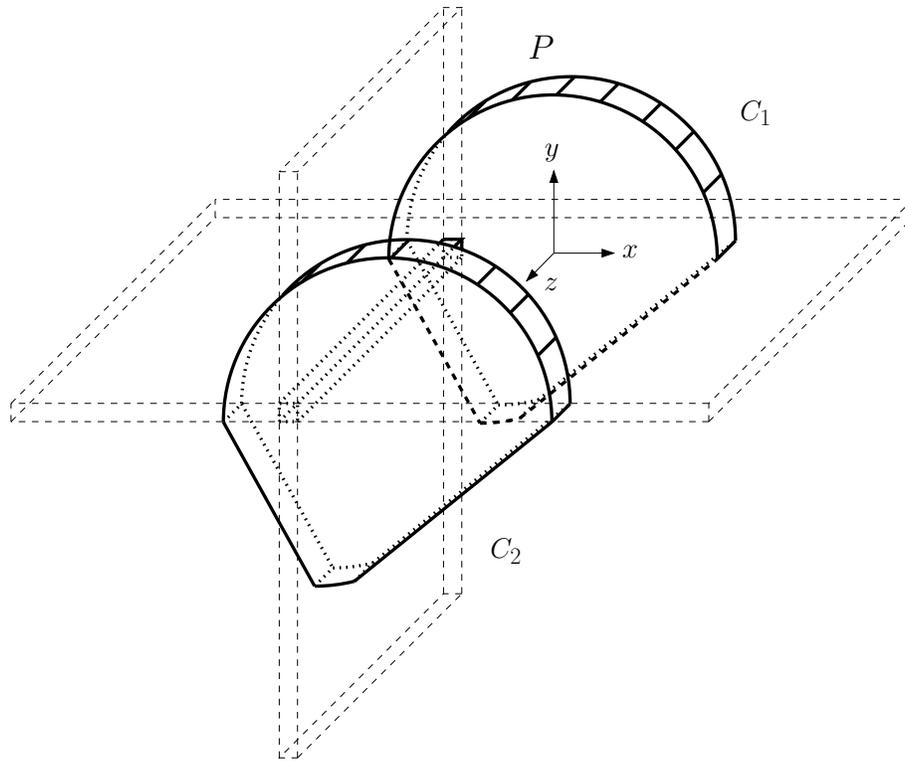


Figure 9: Example of a polyhedron with  $\Omega(n^4)$  combinatorially distinct valid (robust) casting lines is shown in bold. The two sets of planes containing all of the black regions introduced by the box joining  $C_1$  and  $C_2$  are shown as dashed boxes.

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