Higher-Order Triangular-Distance Delaunay Graphs: Graph-Theoretical Properties

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Abstract

We consider an extension of the triangular-distance Delaunay graphs (TD-Delaunay) on a set P of points in general position in the plane. In TD-Delaunay, the convex distance is defined by a fixed-oriented equilateral triangle $\bigtriangledown$, and there is an edge between two points in P if and only if there is an empty homothet of $\bigtriangledown$ having the two points on its boundary. We consider higher-order triangular-distance Delaunay graphs, namely k-TD, which contains an edge between two points if the interior of the smallest homothet of $\bigtriangledown$ having the two points on its boundary contains at most k points of P. We consider the connectivity, Hamiltonicity and perfect-matching admissibility of k-TD. Finally we consider the problem of blocking the edges of k-TD.

1 Introduction

The *triangular-distance Delaunay graph* of a point set P in the plane, TD-Delaunay for short, was introduced by Chew [12]. A TD-Delaunay is a graph whose convex distance function is defined by a fixed-oriented equilateral triangle. Let $\bigtriangledown$ be a downward equilateral triangle whose barycenter is the origin and one of whose vertices is on the negative y-axis. A *homothet* of $\bigtriangledown$ is obtained by scaling $\bigtriangledown$ with respect to the origin by some factor $\mu \geq 0$, followed by a translation to a point b in the plane: $b + \mu\bigtriangledown = \{b + \mu a : a \in \bigtriangledown\}$. In the TD-Delaunay graph of P, there is a straight-line edge between two points p and q if and only if there exists a homothet of $\bigtriangledown$ having p and q on its boundary and whose interior does not contain any point of P. In other words, (p, q) is an edge of TD-Delaunay graph if and only if there exists an empty downward equilateral triangle having p and q on its boundary. In this case, we say that the edge (p, q) has the *empty triangle property*.

We say that P is in general position if the line passing through any two points from P does not make angles $0°$, $60°$, and $120°$ with horizontal. In this paper we consider point sets in general position and our results assume this pre-condition. If P is in general position, then the TD-Delaunay graph is a planar graph, see [7]. We define $t(p, q)$ as the smallest homothet of $\bigtriangledown$ having p and q on its boundary. See Figure 1(a). Note that $t(p, q)$ has one of p and q at a vertex, and the other one on the opposite side. Thus,

Observation 1. *Each side of $t(p, q)$ contains either p or q.*

A graph G is *connected* if there is a path between any pair of vertices in G. Moreover, G is *k-connected* if there does not exist a set of at most $k - 1$ vertices whose removal disconnects G. In case $k = 2$, G is called *biconnected*. In other words a graph G is biconnected iff there is

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a simple cycle between any pair of its vertices. A *matching* in G is a set of edges in G without common vertices. A *perfect matching* is a matching which matches all the vertices of G. A *Hamiltonian cycle* in G is a cycle (i.e., closed loop) through G that visits each vertex of G exactly once. For $H \subseteq G$ we denote the *bottleneck* of H, i.e., the length of the longest edge in H, by $\lambda(H)$.

Let $K_n(P)$ be a complete edge-weighted geometric graph on a point set P which contains a straight-line edge between any pair of points in P. For an edge (p, q) in $K_n(P)$ let $w(p, q)$ denote the weight of (p, q). A *bottleneck matching* (resp. *bottleneck Hamiltonian cycle*) in P is defined to be a perfect matching (resp. Hamiltonian cycle) in $K_n(P)$, in which the weight of the maximum-weight edge is minimized. A *bottleneck biconnected spanning subgraph* of P is a spanning subgraph, $G(P)$, of $K_n(P)$ which is biconnected and the weight of the longest edge in $G(P)$ is minimized.

A tight lower bound on the size of a maximum matching in a TD-Delaunay graph, i.e. 0-TD, is presented in [4]. In this paper we study higher-order TD-Delaunay graphs. The *order-k TD-Delaunay graph* of a point set P, denoted by k-TD, is a geometric graph which has an edge (p, q) iff the interior of $t(p, q)$ contains at most k points of P; see Figure 1(b). The standard TD-Delaunay graph corresponds to 0-TD. We consider graph-theoretic properties of higher-order TD-Delaunay graphs, such as connectivity, Hamiltonicity, and perfect-matching admissibility. We also consider the problem of blocking TD-Delaunay graphs.

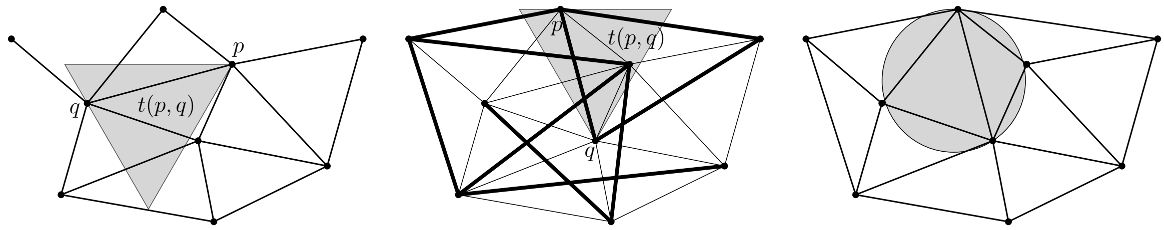


Figure 1: (a) Triangular-distance Delaunay graph (0-TD), (b) 1-TD graph, the light edges belong to 0-TD as well, and (c) Delaunay triangulation.

1.1 Previous Work

A *Delaunay triangulation* (DT) of P (which does not have any four co-circular points) is a graph whose distance function is defined by a fixed circle $\bigcirc$ centered at the origin. DT has an edge between two points p and q iff there exists a homothet of $\bigcirc$ having p and q on its boundary and whose interior does not contain any point of P; see Figure 1(c). In this case the edge (p, q) is said to have the *empty circle property*. The *order-k Delaunay Graph* on P, denoted by k-DG, is defined to have an edge (p, q) iff there exists a homothet of $\bigcirc$ having p and q on its boundary and whose interior contains at most k points of P. The standard Delaunay triangulation corresponds to 0-DG.

For each pair of points $p, q \in P$ let $D[p, q]$ be the closed disk having pq as diameter. A *Gabriel Graph* on P is a geometric graph which has an edge between two points p and q iff $D[p, q]$ does not contain any point of $P \setminus \{p, q\}$. The *order-k Gabriel Graph* on P, denoted by k-GG, is defined to have an edge (p, q) iff $D[p, q]$ contains at most k points of $P \setminus \{p, q\}$.

For each pair of points $p, q \in P$, let $L(p, q)$ be the intersection of the two open disks with radius $|pq|$ centered at p and q, where $|pq|$ is the Euclidean distance between p and q. A *Relative Neighborhood Graph* on P is a geometric graph which has an edge between two points p and q iff $L(p, q)$ does not contain any point of P. The *order-k Relative Neighborhood Graph* on P, denoted by k-RNG, is defined to have an edge (p, q) iff $L(p, q)$ contains at most k points of P. It is obvious that for a fixed point set, k-RNG is a subgraph of k-GG, and k-GG is a subgraph of k-DG.

The problem of determining whether an order-k geometric graph always has a (bottleneck) perfect matching or a (bottleneck) Hamiltonian cycle is of interest. In order to show the importance of this problem we provide the following example. Gabow and Tarjan [15] showed that a bottleneck matching of maximum cardinality in a graph can be computed in $O(m \cdot (n \log n)^{0.5})$ time, where m is the number of edges in the graph. Using their algorithm, a bottleneck perfect matching of a point set can be computed in $O(n^2 \cdot (n \log n)^{0.5})$ time; note that the complete graph on n points has $\Theta(n^2)$ edges. Chang et al. [11] showed that a bottleneck perfect matching of a point set is contained in 16-DG; this graph has $\Theta(n)$ edges and can be computed in $O(n \log n)$ time. Thus, by running the algorithm of Gabow and Tarjan on 16-DG, a bottleneck perfect matching of a point set can be computed in $O(n \cdot (n \log n)^{0.5})$ time.

If for each edge (p, q) in $K_n(P)$, $w(p, q)$ is equal the Euclidean distance between p and q, then Chang et al. [9, 10, 11] proved that a bottleneck biconnected spanning graph, bottleneck perfect matching, and bottleneck Hamiltonian cycle of P are contained in 1-RNG, 16-RNG, 19-RNG, respectively. This implies that 16-RNG has a perfect matching and 19-RNG is Hamiltonian. Since k-RNG is a subgraph of k-GG, the same results hold for 16-GG and 19-GG. It is known that k-GG is $(k + 1)$-connected [8] and 10-GG (and hence 10-DG) is Hamiltonian [16]. Dillencourt showed that every Delaunay triangulation (0-DG) admits a perfect matching [14] but it can fail to be Hamiltonian [13].

Given a geometric graph $G(P)$ on a set P of n points, we say that a set K of points *blocks* $G(P)$ if in $G(P \cup K)$ there is no edge connecting two points in P. Actually P is an independent set in $G(P \cup K)$. Aichholzer et al. [2] considered the problem of blocking the Delaunay triangulation (i.e. 0-DG) for a given point set P in which no four points are co-circular. They show that $\frac{3n}{2}$ points are sufficient to block 0-DG and $n - 1$ points are necessary. To block a Gabriel graph, $n - 1$ points are sufficient [3].

In a companion paper, we considered the matching and blocking problems in higher-order Gabriel graphs. We showed that 10-GG contains a Euclidean bottleneck matching and 8-GG may not have any. As for maximum matching, we proved a tight lower bound of $\frac{n-1}{4}$ in 0-GG. We also showed that 1-GG has a matching of size at least $\frac{2(n-1)}{5}$ and 2-GG has a perfect matching (when n is even). In addition, we showed that $\lceil \frac{n-1}{3} \rceil$ points are necessary to block 0-TD and this bound is tight.

1.2 Our Results

We consider some graph-theoretical properties of higher-order triangular distance Delaunay graphs on a given set P of n points in general position in the plane. We show for which values of k, k-TD contains a bottleneck biconnected spanning graph, a bottleneck Hamiltonian cycle, and a (bottleneck) perfect-matching; for the bottleneck structures we assume that the weight of any edge (p, q) in $K_n(P)$ is equal to the area of the smallest homothet of $\bigtriangledown$ having p and q on its boundary. In Section 3 we prove that every k-TD graph is $(k+1)$-connected. In addition we show that a bottleneck biconnected spanning graph of P is contained in 1-TD. Using a similar approach as in [1, 9], in Section 4 we show that a bottleneck Hamiltonian cycle of P is contained in 7-TD. We also show a configuration of a point set P such that 5-TD fails to have a bottleneck Hamiltonian cycle. In Section 5 we prove that a bottleneck perfect matching of P is contained in 6-TD, and we show that for some point set P, 5-TD does not have a bottleneck perfect matching. In Section 5.2 we prove that 2-TD has a perfect matching and 1-TD has a matching of size at least $\frac{2(n-1)}{5}$. In Section 6 we consider the problem of blocking k-TD. We show that at least $\lceil \frac{n-1}{2} \rceil$ points are necessary and $n-1$ points are sufficient to block a 0-TD. The open problems and concluding remarks are presented in Section 7.

2 Preliminaries

Bonichon et al. [6] showed that the half-Θ_6 graph of a point set P in the plane is equal to the TD-Delaunay graph of P. They also showed that every plane triangulation is TD-Delaunay realizable.

The half-Θ_6 graph (or equivalently a TD-Delaunay graph) on a point set P can be constructed in the following way. For each point p in P, let l_p be the horizontal line through p. Define l_p^γ as the line obtained by rotating l_p by γ-degrees in counter-clockwise direction around p. Actually $l_p^0 = l_p$. Consider three lines l_p^0, l_p^{60}, and l_p^{120} which partition the plane into six disjoint cones with apex p. Let $C_p^1, \dots, C_p^6$ be the cones in counter-clockwise order around p as shown in Figure 2. C_p^1, C_p^3, C_p^5 will be referred to as *odd cones*, and C_p^2, C_p^4, C_p^6 will be referred to as *even cones*. For each even cone C_p^i, connect p to the "nearest" point q in C_p^i. The *distance* between p and q, $d(p,q)$, is defined as the Euclidean distance between p and the orthogonal projection of q onto the bisector of C_p^i. See Figure 2. The resulting graph is the half-Θ_6 graph which is defined by even cones [6]. Moreover, the resulting graph is the TD-Delaunay graph defined with respect to homothets of $\bigtriangledown$. By considering the odd cones, another half-Θ_6 graph is obtained. The well-known Θ_6 graph is the union of half-Θ_6 graphs defined by odd and even cones.

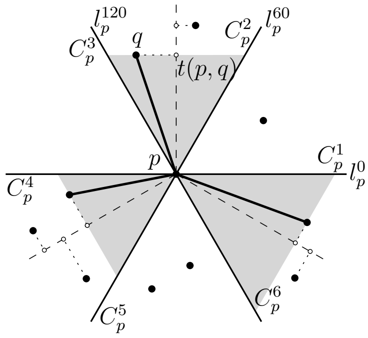


Figure 2: The construction of the TD-Delaunay graph.

Recall that $t(p,q)$ is the smallest homothet of $\bigtriangledown$ having p and q on its boundary. In other words, $t(p,q)$ is the smallest downward equilateral triangle through p and q. Similarly we define $t'(p,q)$ as the smallest upward equilateral triangle having p and q on its boundary. It is obvious that the even cones correspond to downward triangles and odd cones correspond to upward triangles. We define an order on the equilateral triangles: for each two equilateral triangles t_1 and t_2 we say that $t_1 \prec t_2$ if the area of t_1 is less than the area of t_2. Since the area of $t(p,q)$ is directly related to $d(p,q)$,

$$d(p,q) < d(r,s) \quad \text{if and only if} \quad t(p,q) \prec t(r,s).$$

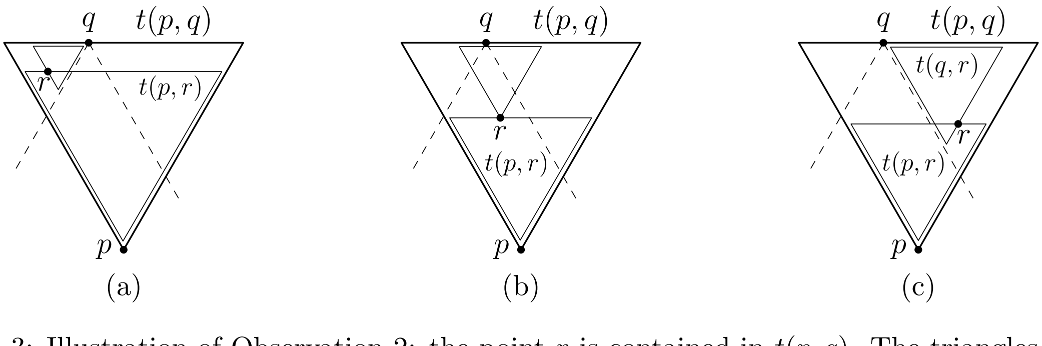


Figure 3: Illustration of Observation 2: the point r is contained in $t(p,q)$. The triangles $t(p,r)$ and $t(q,r)$ are inside $t(p,q)$.

For a set $\{t_1, \dots, t_m\}$ of equilateral triangles we define $\max\{t_1, \dots, t_m\}$ to be the triangle with the largest area. As shown in Figure 3 we have the following observation:

Observation 2. *If $t(p,q)$ contains a point r, then $t(p,r)$ and $t(q,r)$ are contained in $t(p,q)$.*

As a direct consequence of Observation 2, if a point r is contained in $t(p,q)$, then $\max\{t(p,r), t(q,r)\} \prec t(p,q)$. It is obvious that,

Observation 3. *For each two points $p, q \in P$, the area of $t(p,q)$ is equal to the area of $t'(p,q)$.*

Thus, we define $X(p,q)$ as a regular hexagon centered at p which has q on its boundary, and its sides are parallel to l_p^0, l_p^{60}, and l_p^{120}.

Observation 4. *If $X(p,q)$ contains a point r, then $t(p,r) \prec t(p,q)$.*

We construct k-TD as follows. For each point $p \in P$, imagine the six cones having their apex at p, as described earlier. Then connect p to its $(k+1)$ nearest neighbors in each even cone around p. For each edge (p,q) in k-TD we define its *weight*, $w(p,q)$, to be equal to the area of $t(p,q)$. The resulting graphs is k-TD, which has $O(kn)$ edges. The k-TD can be constructed in $O(n \log n + kn \log\log n)$-time, using the algorithm introduced by Lukovszki [17] for computing fault tolerant spanners.

For a graph $G = (V,E)$ and $K \subseteq V$, let $G - K$ be the subgraph obtained from G by removing the vertices in K, and let $o(G-K)$ be the number of odd components in $G-K$. The following theorem by Tutte [18] gives a characterization of the graphs which have a perfect matching:

Theorem 1 (Tutte [18])**.** *G has a perfect matching if and only if $o(G-K) \leq |K|$ for all $K \subseteq V$.*

Berge [5] extended Tutte's theorem to a formula (known as Tutte-Berge formula) for the maximum size of a matching in a graph. In a graph G, the *deficiency*, $\text{def}_G(K)$, is $o(G-K)-|K|$. Let $\text{def}(G) = \max_{K\subseteq V} \text{def}_G(K)$.

Theorem 2 (Tutte-Berge formula; Berge [5])**.** *The size of a maximum matching in G is*

$$\frac{1}{2}(n - \text{def}(G)).$$

For an edge-weighted graph G we define the *weight sequence* of G, $\text{WS}(G)$, as the sequence containing the weights of the edges of G in non-increasing order. For two graphs G_1 and G_2 we say that $\text{WS}(G_1) \prec \text{WS}(G_2)$ if $\text{WS}(G_1)$ is lexicographically smaller than $\text{WS}(G_2)$. A graph G_1 is said to be less than a graph G_2 if $\text{WS}(G_1) \prec \text{WS}(G_2)$.

3 Connectivity

In this section we consider the connectivity of higher-order triangular-distance Delaunay graphs.

3.1 $(k+1)$-connectivity

For a set P of points in the plane, the TD-Delaunay graph, i.e., 0-TD, is not necessarily a triangulation [12], but it is connected and internally triangulated [4], i.e., all internal faces are triangles. As shown in Figure 1(a), 0-TD may not be biconnected. As a warm up exercise we show that every k-TD is $(k+1)$-connected.

Theorem 3. *For every point set P in general position in the plane, k-TD is $(k+1)$-connected. In addition, for every k, there exists a point set P such that k-TD is not $(k+2)$-connected.*

Proof. We prove the first part of this theorem by contradiction. Let K be the set of (at most) k vertices removed from k-TD, and let $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$, where $m > 1$, be the resulting maximal connected components. Let T be the set of all triangles defined by any pair of points belonging to different components, i.e., $T = \{t(a, b) : a \in C_i, b \in C_j, i \neq j\}$. Consider the smallest triangle $t_{min} \in T$. Assume that t_{min} is defined by two points a and b, i.e., $t_{min} = t(a, b)$, where $a \in C_i$, $b \in C_j$, and $i \neq j$.

Claim 1: t_{min} does not contain any point of $P \setminus K$ in its interior. By contradiction, suppose that t_{min} contains a point $c \in P \setminus K$ in its interior. Three cases arise: (i) $c \in C_i$, (ii) $c \in C_j$, (iii) $c \in C_l$, where $l \neq i$ and $l \neq j$. In case (i) the triangle $t(c, b)$ between C_i and C_j is contained in $t(a, b)$. In case (ii) the triangle $t(a, c)$ between C_i and C_j is contained in $t(a, b)$. In case (iii) both triangles $t(a, c)$ and $t(c, b)$ are contained in $t(a, b)$. All cases contradict the minimality of $t(a, b) = t_{min}$. Thus, t_{min} contains no point of $P \setminus K$ in its interior, proving Claim 1.

By Claim 1, $t_{min} = t(a, b)$ may only contain points of K. Since $|K| \leq k$, there must be an edge between a and b in k-TD. This contradicts that a and b belong to different components C_i and C_j in $\mathcal{C}$. Therefore, k-TD is $(k + 1)$-connected.

We present a constructive proof for the second part of theorem. Let $P = A \cup B \cup K$, where $|A|, |B| \geq 1$ and $|K| = k + 1$. Place the points of A in the plane. Let $C^4_A = \bigcap_{p \in A} C^4_p$. Place the points of K in C^4_A. Let $C^4_K = \bigcap_{p \in K} C^4_p$. Place the points of B in C^4_K. Consider any pair (a, b) of points where $a \in A$ and $b \in B$. It is obvious that any path between a and b in k-TD goes through the vertices in K. Thus by removing the vertices in K, a and b become disconnected. Therefore, k-TD of P is not $(k + 2)$-connected. □

3.2 Bottleneck Biconnected Spanning Graph

As shown in Figure 1(a), 0-TD may not be biconnected. By Theorem 3, 1-TD is biconnected. In this section we show that a bottleneck biconnected spanning graph of P is contained in 1-TD.

Theorem 4. *For every point set P in general position in the plane,* 1-*TD contains a bottleneck biconnected spanning graph of P.*

Proof. Let $\mathcal{G}$ be the set of all biconnected spanning graphs with vertex set P. We define a total order on the elements of $\mathcal{G}$ by their weight sequence. If two elements have the same weight sequence, we break the ties arbitrarily to get a total order. Let $G^* = (P, E)$ be a graph in $\mathcal{G}$ with minimal weight sequence. Clearly, G^* is a bottleneck biconnected spanning graph of P. We will show that all edges of G^* are in 1-TD. By contradiction suppose that some edges in E do not belong to 1-TD, and let $e = (a, b)$ be the longest one (by the area of the triangle $t(a, b)$). If the graph $G^* - \{e\}$ is biconnected, then by removing e, we obtain a biconnected spanning graph G with $\mathrm{WS}(G) \prec \mathrm{WS}(G^*)$; this contradicts the minimality of G^*. Thus, there is a pair $\{p, q\}$ of points such that any cycle between p and q in G^* goes through e. Since $(a, b) \notin$ 1-TD, $t(a, b)$ contains at least two points of P, say x and y. Let G be the graph obtained from G^* by removing the edge (a, b) and adding the edges (a, x), (b, x), (a, y), (b, y). We show that in G there is a cycle C between p and q which does not go through e. Consider a cycle C^* in G^* between two points p and q (which goes through e). If none of x and y belong to C^*, then $C = (C^* - \{(a, b)\}) \cup \{(a, x), (b, x)\}$ is a cycle in G between p and q. If one of x or y, say x, belongs to C^*, then $C = (C^* - \{(a, b)\}) \cup \{(a, y), (b, y)\}$ is a cycle in G between p and q. If both x and y belong to C^*, w.l.o.g. assume that x is between b and y in the path $C^* - \{(a, b)\}$. Consider the partition of C^* into four parts: (a) edge (a, b), (b) path δ_{bx} between b and x, (c) path δ_{xy} between x and y, and (d) path δ_{ya} between y and a. There are four cases:

1. None of p and q are on δ_{xy}. Let $C = \delta_{bx} \cup \delta_{ya} \cup \{(a, x), (b, y)\}$.

2. Both p and q are on δ_{xy}. Let $C = \delta_{xy} \cup \{(a,x),(a,y)\}$.

3. One of p,q is on δ_{xy} while the other is on δ_{bx}. Let $C = \delta_{bx} \cup \delta_{xy} \cup \{(b,y)\}$.

4. One of p,q is on δ_{xy} while the other is on δ_{ya}. Let $C = \delta_{xy} \cup \delta_{ya} \cup \{(a,x)\}$.

In all cases, C is a cycle in G between p and q. Thus, between any pair of points in G there exists a cycle, and hence G is biconnected. Since x and y are inside $t(a,b)$, by Observation 2, $\max\{t(a,x),t(a,y),t(b,x),t(b,y)\} \prec t(a,b)$. Therefore, $\text{WS}(G) \prec \text{WS}(G^*)$; this contradicts the minimality of G^*. □

4 Hamiltonicity

In this section we show that 7-TD contains a bottleneck Hamiltonian cycle. In addition, we will show that for some point sets, 5-TD does not contain any bottleneck Hamiltonian cycle.

Theorem 5. *For every point set P in general position in the plane, 7-TD contains a bottleneck Hamiltonian cycle.*

Proof. Let $\mathcal{H}$ be the set of all Hamiltonian cycles through the points of P. Define a total order on the elements of $\mathcal{H}$ by their weight sequence. If two elements have exactly the same weight sequence, break ties arbitrarily to get a total order. Let $H^* = a_0, a_1, \dots, a_{n-1}, a_0$ be a cycle in $\mathcal{H}$ with minimal weight sequence. It is obvious that H^* is a bottleneck Hamiltonian cycle of P. We will show that all the edges of H^* are in 7-TD. Consider any edge $e = (a_i, a_{i+1})$ in H^* and let $t(a_i, a_{i+1})$ be the triangle corresponding to e (all the index manipulations are modulo n).

Claim 1: None of the edges of H^* can be completely in the interior $t(a_i, a_{i+1})$. Suppose there is an edge $f = (a_j, a_{j+1})$ inside $t(a_i, a_{i+1})$. Let H be a cycle obtained from H^* by deleting e and f, and adding (a_i, a_j) and (a_{i+1}, a_{j+1}). By Observation 2, $t(a_i, a_{i+1}) \succ \max\{t(a_i, a_j), t(a_{i+1}, a_{j+1})\}$, and hence $\text{WS}(H) \prec \text{WS}(H^*)$. This contradicts the minimality of H^*.

Therefore, we may assume that no edge of H^* lies completely inside $t(a_i, a_{i+1})$. Suppose there are w points of P inside $t(a_i, a_{i+1})$. Let $U = u_1, u_2, \dots, u_w$ represent these points indexed in the order we would encounter them on H^* starting from a_i. Let $R = \{r_1, r_2, \dots, r_w\}$ represent the vertices where r_i is the vertex succeeding u_i in the cycle. All the vertices in R, probably except r_w, are different from a_i (and a_{i+1}). Without loss of generality assume that $a_i \in C^4_{a_{i+1}}$, and $t(a_i, a_{i+1})$ is anchored at a_{i+1}, as shown in Figure 4.

Claim 2: For each $r_j \in R$, $t(r_j, a_{i+1}) \succeq \max\{t(a_i, a_{i+1}), t(u_j, r_j)\}$. Suppose there is a point $r_j \in R$ such that $t(r_j, a_{i+1}) \prec \max\{t(a_i, a_{i+1}), t(u_j, r_j)\}$. Construct a new cycle H by removing the edges (u_j, r_j), (a_i, a_{i+1}) and adding the edges (a_{i+1}, r_j) and (a_i, u_j). Since the two new edges have length strictly less than $\max\{t(a_i, a_{i+1}), t(u_j, r_j)\}$, $\text{WS}(H) \prec \text{WS}(H^*)$; which is a contradiction.

Claim 3: For each $r_j, r_k \in R$, $t(r_j, r_k) \succeq \max\{t(a_i, a_{i+1}), t(u_j, r_j), t(u_k, r_k)\}$. Suppose there is a pair r_j and r_k such that $t(r_j, r_k) \prec \max\{t(a_i, a_{i+1}), t(u_j, r_j), d(u_k, r_k)\}$. Construct a cycle H from H^* by first deleting (u_j, r_j), (u_k, r_k), (a_i, a_{i+1}). This results in three paths. One of the paths must contain both a_i and either r_j or r_k. W.l.o.g. suppose that a_i and r_k are on the same path. Add the edges (a_i, u_j), (a_{i+1}, u_k), (r_j, r_k). Since $\max\{t(u_j, r_j), t(u_k, r_k), d(a_i, a_{i+1})\} \succ \max\{t(a_i, u_j), t(a_{i+1}, u_k), t(r_j, r_k)\}$, $\text{WS}(H) \prec \text{WS}(H^*)$; we get a contradiction.

We use Claim 2 and Claim 3 to show that the size of R is at most seven, and consequently $w \le 7$. Consider the lines $l^0_{a_{i+1}}$, $l^{60}_{a_{i+1}}$, $l^{120}_{a_{i+1}}$, and $l^{120}_{a_i}$ as shown in Figure 4. Let l_1 and l_2 be

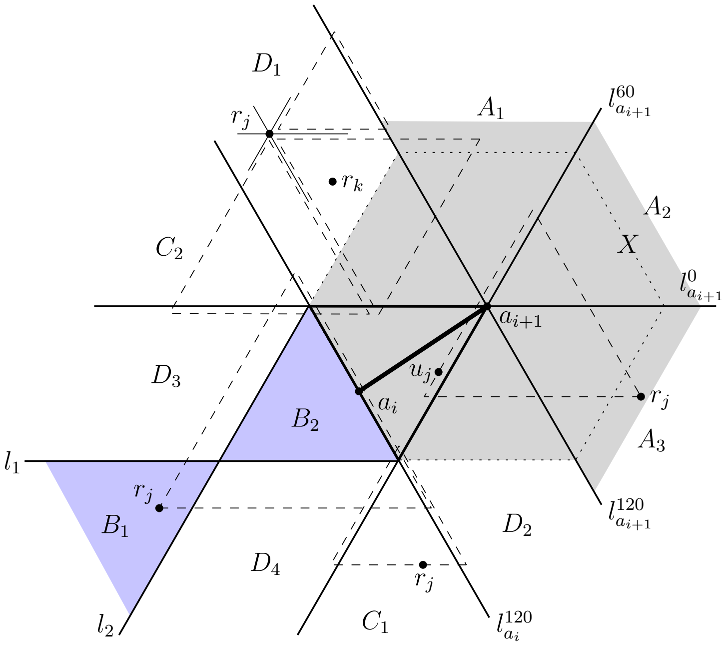


Figure 4: Illustration of Theorem 5.

the rays starting at the corners of $t(a_i, a_{i+1})$ opposite to a_{i+1} and parallel to $l^0_{a_{i+1}}$ and $l^{60}_{a_{i+1}}$ respectively. These lines and rays partition the plane into 12 regions, as shown in Figure 4. We will show that each of the regions D_1, D_2, D_3, D_4, C_1, C_2, and $B = B_1 \cup B_2$ contains at most one point of R, and the other regions do not contain any point of R. Consider the hexagon $X(a_{i+1}, a_i)$. By Claim 2 and Observation 4, no point of R can be inside $X(a_{i+1}, a_i)$. Moreover, no point of R can be inside the cones A_1, A_2, or A_3, because if $r_j \in \{A_1 \cup A_2 \cup A_3\}$, the (upward) triangle $t'(u_j, r_j)$ contains a_{i+1}. Then by Observation 4, $t(r_j, a_{i+1}) \prec t(u_j, r_j)$; which contradicts Claim 2.

We show that each of the regions D_1, D_2, D_3, D_4 contains at most one point of R. Consider the region D_1; by similar reasoning we can prove the claim for D_2, D_3, D_4. Using contradiction, let r_j and r_k be two points in D_1, and w.l.o.g. assume that r_j is the farthest to $l^{60}_{a_{i+1}}$. Then r_k can lie inside any of the cones $C^1_{r_j}$, $C^5_{r_j}$, and $C^6_{r_j}$ (but not in X). If $r_k \in C^1_{r_j}$, then $t'(r_j, r_k)$ is smaller than $t'(a_i, a_{i+1})$ which means that $t(r_j, r_k) \prec t(a_i, a_{i+1})$. If $r_k \in C^5_{r_j}$, then $t'(u_j, r_j)$ contains r_k, that is $t(r_j, r_k) \prec t(u_j, r_j)$. If $r_k \in C^6_{r_j}$, then $t(u_j, r_j)$ contains r_k, that is $t(r_j, r_k) \prec t(u_j, r_j)$. All cases contradict Claim 3.

Now consider the region C_1 (or C_2). By contradiction assume that it contains two points r_j and r_k. Let r_j be the farthest from $l^0_{a_{i+1}}$. It is obvious that $t'(u_j, r_j)$ contains r_k, that is $t(r_j, r_k) \prec t(u_j, r_j)$; which contradicts Claim 3.

Consider the region $B = B_1 \cup B_2$. Note that it is possible for r_j or r_k to be a_i. If both r_j and r_k belong to B_2, then $t'(r_j, r_k)$ is smaller that $t(a_i, a_{i+1})$. If $r_j \in B_1$ and $r_k \in B_2$, then $t'(u_j, r_j)$ contains r_k, and hence $t(r_j, r_k) \prec t(u_j, r_j)$. If both r_j and r_k belong to B_1, let r_j be the farthest from $l^{120}_{a_i}$. Clearly, $t(u_j, r_j)$ contains r_k and hence $t(r_j, r_k) \prec t(u_j, r_j)$. All cases contradict Claim 3.

Therefore, any of the regions D_1, D_2, D_3, D_4, C_1, C_2, and $B = B_1 \cup B_2$ contains at most one point of R. Thus, $|R| \leq 7$ and $w \leq 7$, and $t(a_i, a_{i+1})$ contains at most 7 points of P. Therefore, $e = (a_i, a_{i+1})$ is an edge of 7-TD. □

As a direct consequence of Theorem 5 we have shown that:

Corollary 1. *7-TD is Hamiltonian.*

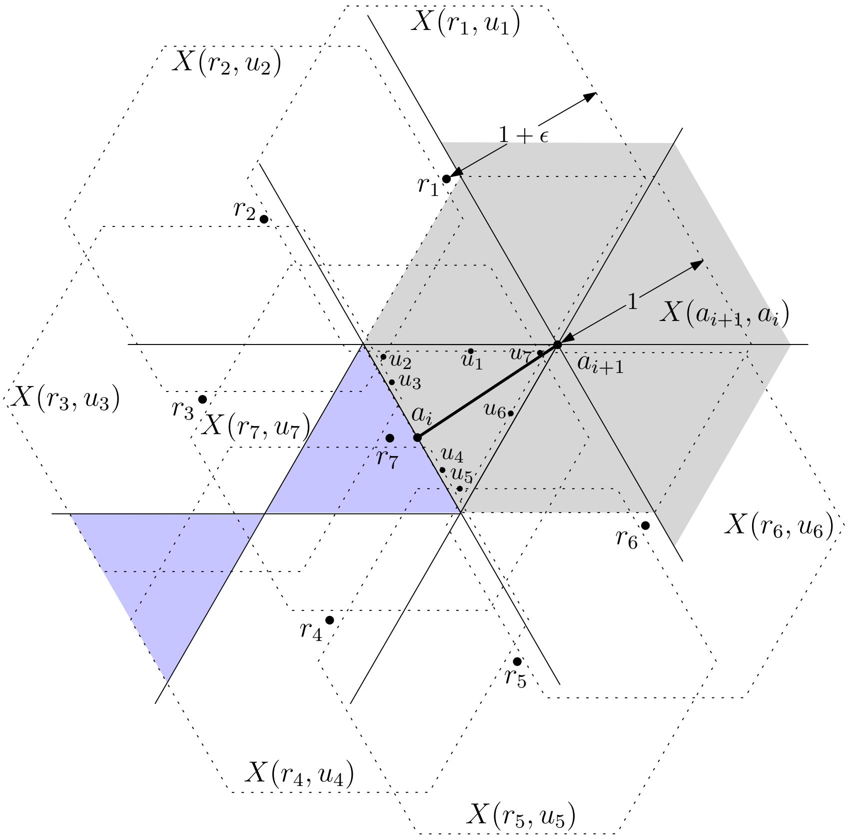


Figure 5: $t(a_i, a_{i+1})$ contains 7 points while the conditions in the proof of Theorem 5 hold.

An interesting question is to determine if k-TD contains a bottleneck Hamiltonian cycle for $k < 7$. Figure 5 shows a configuration where $t(a_i, a_{i+1})$ contains 7 points while the conditions of Claim 1, Claim 2, and Claim 3 in the proof of Theorem 5 hold. In Figure 5, $d(a_i, a_{i+1}) = 1$, $d(r_i, u_i) = 1 + \epsilon$, $d(r_i, r_j) > 1 + \epsilon$, $d(r_i, a_{i+1}) > 1 + \epsilon$ for $i, j = 1, \ldots 7$ and $i \neq j$.

Theorem 6. *There exists an arbitrary large point set such that its* 5*-TD does not contain any bottleneck Hamiltonian cycle.*

Proof. In order to prove the theorem, we provide such a point set. Figure 6 shows a configuration of P with 17 points such that 5-TD does not contain a bottleneck Hamiltonian cycle. In Figure 6, $d(a, b) = 1$ and $t(a, b)$ contains 6 points $U = \{u_1, \ldots, u_6\}$. In addition $d(r_i, u_i) = 1 + \epsilon$, $d(r_i, r_j) > 1 + \epsilon$, $d(r_i, b) > 1 + \epsilon$ for $i, j = 1, \ldots 6$ and $i \neq j$. Let $R = \{t_1, t_2, t_3, r_1, \ldots, r_6\}$. The dashed hexagons are centered at a and b and have diameter 1. The dotted hexagons are centered at vertices in R and have diameter $1 + \epsilon$. Each point in R is connected to its first and second closest points by edges of length $1 + \epsilon$ (the bold edges). Let B be the set of these edges. Let H be a cycle formed by $B \cup \{(u_3, b), (b, a), (a, u_5)\}$, i.e., $H = (u_4, r_4, u_5, r_5, u_6, r_6, t_1, t_2, t_3, r_1, u_1, r_2, u_2, r_3, u_3, a, b, u_4)$. It is obvious that H is a Hamiltonian cycle for P and $\lambda(H) = 1+\epsilon$. Thus, the bottleneck of any bottleneck Hamiltonian cycle for P is at most $1 + \epsilon$. We will show that any bottleneck Hamiltonian cycle for P contains the edge (a, b) which does not belong to 5-TD. By contradiction, let H^* be a bottleneck Hamiltonian cycle which does not contain (a, b). In H^*, b is connected to two vertices b_l and b_r, where $b_l \neq a$ and $b_r \neq a$. Since the distance between b and any vertex in R is strictly bigger than $1 + \epsilon$ and

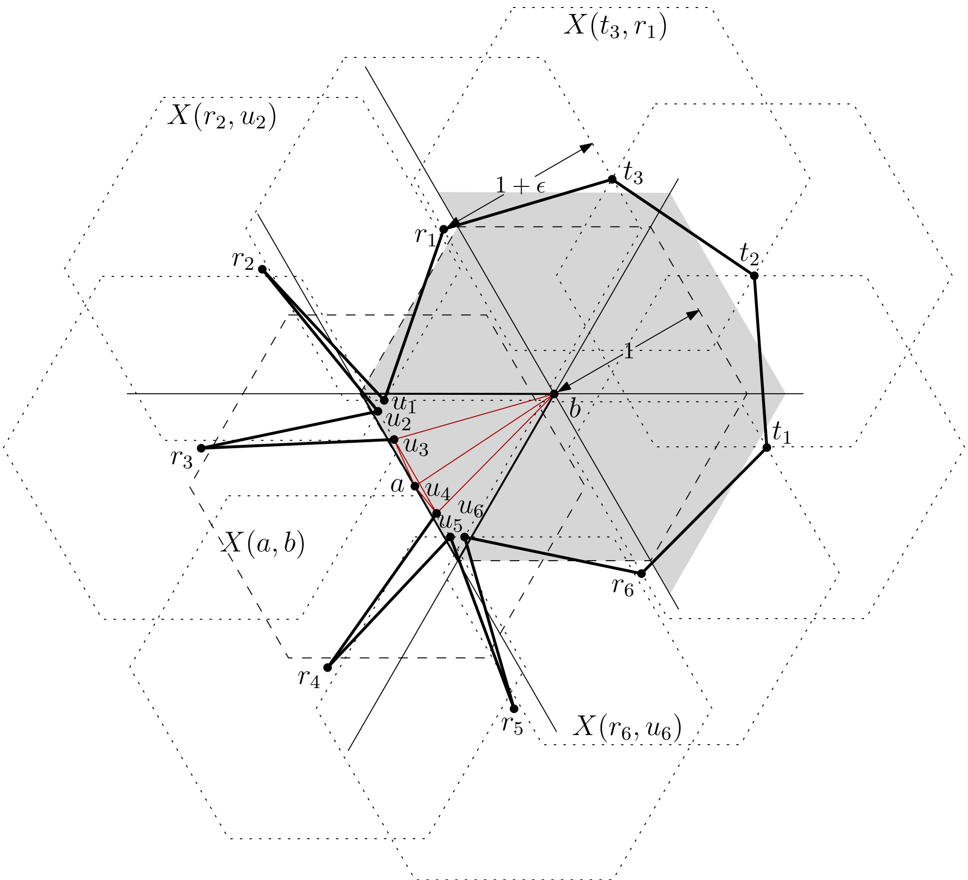


Figure 6: The points $\{r_1, \ldots, r_6, t_1, t_2, t_3\}$ are connected to their first and second closest point (the bold edges). The edge (a, b) should be in any bottleneck Hamiltonian cycle, while $t(a, b)$ contains 6 points.

$\lambda(H^*) \leq 1 + \epsilon$, $b_l \notin R$ and $b_r \notin R$. Thus b_l and b_r belong to U. Let $U' = \{u_1, u_2, u_5, u_6\}$. Consider two cases:

- $b_l \in U'$ or $b_r \in U'$. W.l.o.g. assume that $b_l \in U'$ and $b_l = u_1$. Since u_1 is the first/second closest point of r_1 and r_2, in H^* one of r_1 and r_2 must be connected by an edge e to a point that is farther than its second closet point; e has length strictly greater than $1 + \epsilon$.
- $b_l \notin U'$ and $b_r \notin U'$. Thus, both b_l and b_r belong to $\{u_3, u_4\}$. That is, in H^*, a should be connected to a point c where $c \in R \cup U'$. If $c \in R$ then the edge (a, c) has length more than $1 + \epsilon$. If $c \in U'$, w.l.o.g. assume $c = u_1$; by the same argument as in the previous case, one of r_1 and r_2 must be connected by an edge e to a point that is farther than its second closet point; e has length strictly greater than $1 + \epsilon$.

Since $e \in H^*$, both cases contradicts that $\lambda(H^*) \leq 1+\epsilon$. Therefore, every bottleneck Hamiltonian cycle contains edge (a, b). Since (a, b) is not an edge in 5-TD, a bottleneck Hamiltonian cycle of P is not contained in 5-TD. We can construct larger point sets by adding new points very close to t_2, and at distance at least $1 + 2\epsilon$ from b. $\square$

5 Perfect Matching Admissibility

In this section we consider the matching problem in higher-order triangular-distance Delaunay graphs. In Subsection 5.1 we show that 6-TD contains a bottleneck perfect matching. We also show that for some point sets P, 5-TD does not contain any bottleneck perfect matching. In Subsection 5.2 we prove that every 2-TD has a perfect matching when P has an even number of points, and 1-TD contains a matching of size at least $\frac{2(n-1)}{5}$.

5.1 Bottleneck Perfect Matching

Theorem 7. *For a set P of an even number of points in general position in the plane, 6-TD contains a bottleneck perfect matching.*

Proof. Let $\mathcal{M}$ be the set of all perfect matchings through the points of P. Define a total order on the elements of $\mathcal{M}$ by their weight sequence. If two elements have exactly the same weight sequence, break ties arbitrarily to get a total order. Let $M^* = \{(a_1, b_1), \dots, (a_{\frac{n}{2}}, b_{\frac{n}{2}})\}$ be a perfect matching in $\mathcal{M}$ with minimal weight sequence. It is obvious that M^* is a bottleneck perfect matching for P. We will show that all edges of M^* are in 6-TD. Consider any edge $e = (a_i, b_i)$ in M^* and its corresponding triangle $t(a_i, b_i)$.

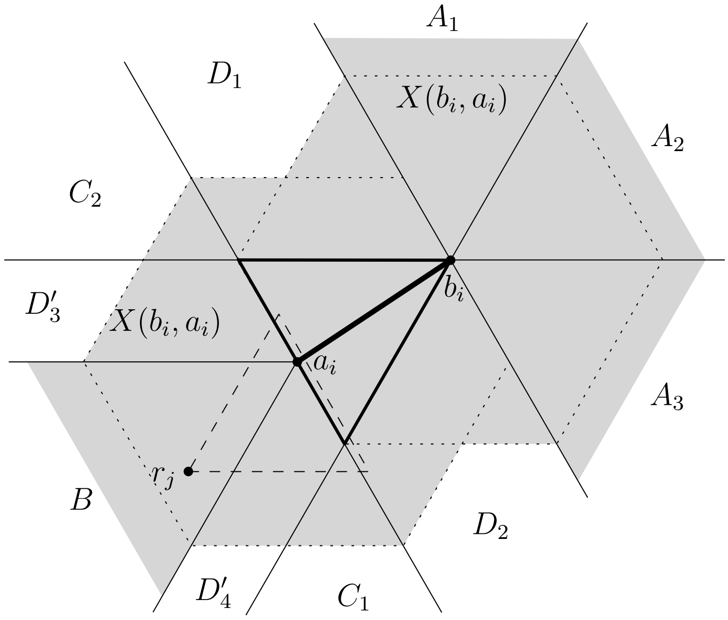


Figure 7: Proof of Theorem 7.

Claim 1: None of the edges of M^* can be inside $t(a_i, b_i)$. Suppose there is an edge $f = (a_j, b_j)$ inside $t(a_i, b_i)$. Let M be a perfect matching obtained from M^* by deleting $\{e, f\}$, and adding $\{(a_i, a_j), (b_i, b_j)\}$. By Observation 2, the two new edges are smaller than the old ones. Thus, $\text{WS}(M) \prec \text{WS}(M^*)$ which contradicts the minimality of M^*.

Therefore, we may assume that no edge of M^* lies completely inside $t(a_i, b_i)$. Suppose there are w points of P inside $t(a_i, b_i)$. Let $U = u_1, u_2, \dots, u_w$ represent the points inside $t(a_i, b_i)$, and $R = r_1, r_2, \dots, r_w$ represent the points where $(r_i, u_i) \in M^*$. W.l.o.g. assume that $a_i \in C^4_{b_i}$, and $t(a_i, b_i)$ is anchored at b_i as shown in Figure 7.

Claim 2: For each $r_j \in R$, $\min\{t(r_j, a_i), t(r_j, b_i)\} \succeq \max\{t(a_i, b_i), t(u_j, r_j)\}$. Otherwise, by a similar argument as in the proof of Claim 2 in Theorem 5 we can either match r_j with a_i or b_i to obtain a smaller matching M; which is a contradiction.

Claim 3: For each pair r_j and r_k of points in R, $t(r_j, r_k) \succeq \max\{t(a_i, b_i), t(r_j, u_j), t(r_k, u_k)\}$. The proof is similar to the proof of Claim 3 in Theorem 5.

Consider Figure 7 which partitions the plane into eleven regions. As a direct consequence of Claim 2, the hexagons $X(b_i, a_i)$ and $X(a_i, b_i)$ do not contain any point of R. By a similar argument as in the proof of Theorem 5, the regions A_1, A_2, A_3 do not contain any point of R. In addition, the region B does not contain any point r_j of R, because otherwise $t'(r_j, u_j)$ contains a_i, that is $t(r_j, a_i) \prec t(u_j, r_j)$ which contradicts Claim 2. As shown in the proof of Theorem 5 each of the regions D_1, D_2, D'_3, D'_4, C_1, and C_2 contains at most one point of R (note that $D'_3 \subset D_3$ and $D'_4 \subset D_4$). Thus, $w \leq 6$, and $t(a_i, b_i)$ contains at most 6 points of P. Therefore, $e = (a_i, b_i)$ is an edge of 6-TD. □

As a direct consequence of Theorem 7 we have shown that:

Corollary 2. *For a set P of even number of points in general position in the plane,* 6*-TD has a perfect matching.*

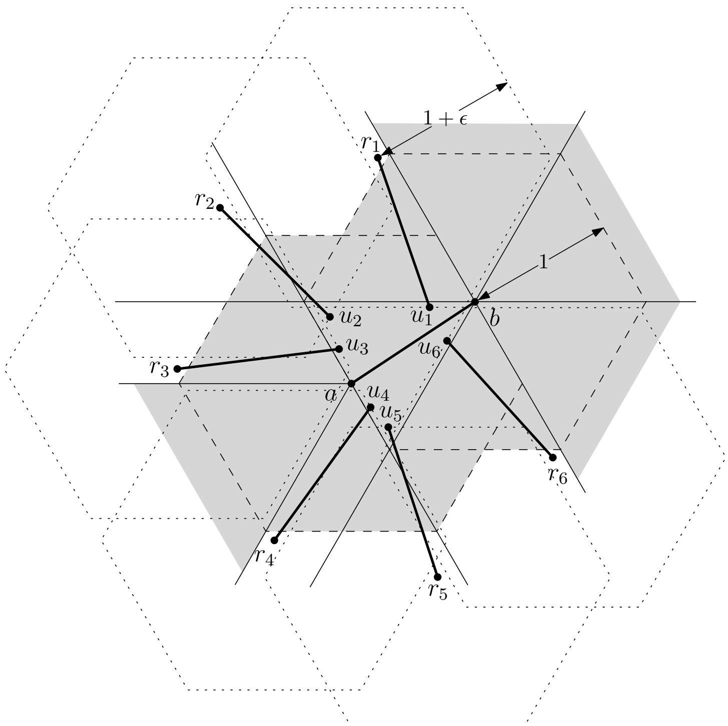


Figure 8: The points $\{r_1, \ldots, r_6\}$ are matched to their closest point. The edge (a, b) should be an edge in any bottleneck perfect matching, while $t(a, b)$ contains 6 points.

In the following theorem, we show that the bound $k = 6$ proved in Theorem 7 is tight.

Theorem 8. *There exists an arbitrarily large point set such that its* 5*-TD does not contain any bottleneck perfect matching.*

Proof. In order to prove the theorem, we provide such a point set. Figure 8 shows a configuration of a set P with 14 points such that $d(a, b) = 1$ and $t(a, b)$ contains six points $U = \{u_1, \ldots, u_6\}$. In addition $d(r_i, u_i) = 1 + \epsilon$, $d(r_i, x) > 1 + \epsilon$ where $x \neq u_i$, for $i = 1, \ldots 6$. Let $R = \{r_1, \ldots, r_6\}$. In Figure 8, the dashed hexagons are centered at a and b, each of diameter 1, and the dotted hexagons centered at vertices in R, each of diameter $1 + \epsilon$. Consider a perfect matching $M = \{(a, b)\} \cup \{(r_i, u_i) : i = 1, \ldots, 6\}$ where each point $r_i \in R$ is matched to its closest point u_i. It is obvious that $\lambda(M) = 1 + \epsilon$, and hence the bottleneck of any bottleneck perfect matching is at most $1+\epsilon$. We will show that any bottleneck perfect matching for P contains the edge (a, b) which does not belong to 5-TD. By contradiction, let M^* be a bottleneck perfect matching which does not contain (a, b). In M^*, b is matched to a point $c \in R \cup U$. If $c \in R$, then $d(b, c) > 1+\epsilon$. If $c \in U$, w.l.o.g. assume $c = u_1$. Thus, in M^* the point r_1 is matched to a point d where $d \neq u_1$. Since u_1 is the unique closest point to r_1 and $d(r_1, u_1) = 1 + \epsilon$, $d(r_1, d) > 1 + \epsilon$. Both cases contradicts that $\lambda(M^*) \leq 1 + \epsilon$. Therefore, every bottleneck perfect matching contains (a, b). Since (a, b) is not an edge in 5-TD, a bottleneck perfect matching of P is not contained in 5-TD. We can construct larger point sets by adding new points—which are within distance $1 + \epsilon$ from each other—at distance at least $1 + 2\epsilon$ from the current point set. □

5.2 Perfect Matching

In [4] the authors proved a tight lower bound of $\lceil \frac{n-1}{3} \rceil$ on the size of a maximum matching in 0-TD. In this section we prove that 1-TD has a matching of size $\frac{2(n-1)}{5}$ and 2-TD has a perfect matching when P has an even number of points.

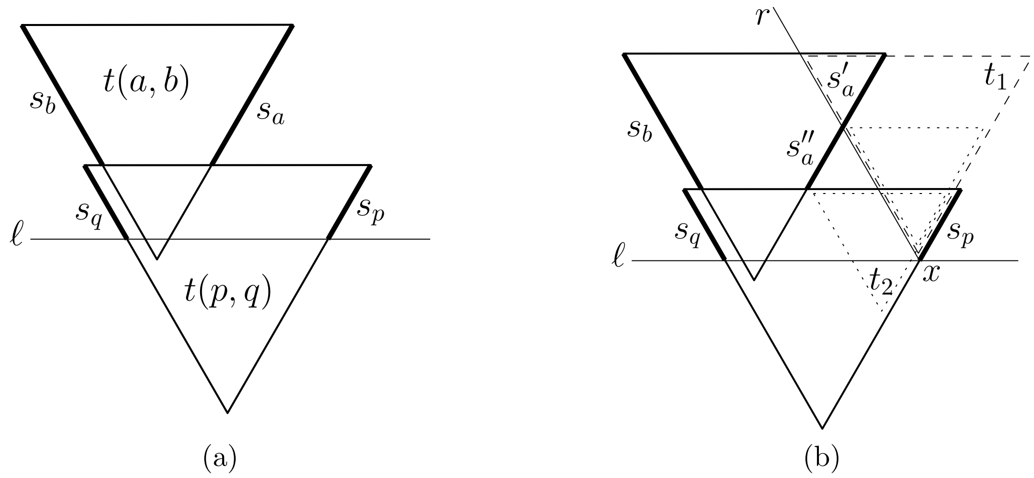


Figure 9: (a) Illustration of Lemma 1, and (b) proof of Lemma 1.

For a triangle $t(a, b)$ through the points a and b, let *top*(a, b), $left(a, b)$, and $right(a, b)$ respectively denote the top, left, and right sides of $t(a, b)$. Refer to Figure 9(a) for the following lemma.

Lemma 1. *Let $t(a, b)$ and $t(p, q)$ intersect a horizontal line ℓ, and $t(a, b)$ intersects top(p, q) in such a way that $t(p, q)$ contains the lowest corner of $t(a, b)$. Let a (resp. p) lie on $right(a, b)$ (resp. $right(p, q)$. If a and b lie above top(p, q), and p and q lie above ℓ, then,* $\max\{t(a, p), t(b, q)\} \prec \max\{t(a, b), t(p, q)\}$.

Proof. Recall that $t(a, b)$ is the smallest downward triangle through a and b. By Observation 1 each side of $t(a, b)$ contains either a or b. In Figure 9(a) the set of potential positions for point a on the boundary of $t(a, b)$ is shown by the line segment s_a; and similarly by s_b, s_p, s_q for b, p, q, respectively. We will show that $t(a, p) \prec \max\{t(a, b), t(p, q)\}$. By similar reasoning we can show that $t(b, q) \prec \max\{t(a, b), t(p, q)\}$. Let x denote the intersection of ℓ and $right(p, q)$. Consider a ray r initiated at x and parallel to $left(p, q)$ which divides s_a into (at most) two parts s'_a and s''_a as shown in Figure 9(b). Two cases may appear:

- $a \in s'_a$. Let t_1 be a downward triangle anchored at x which has its *top* side on the line through *top*(a, b) (the dashed triangle in Figure 9(b)). The top side of t_1 and $t(a, b)$ lie on the same horizontal line. The bottommost corner of t_1 is on ℓ while the bottommost corner of $t(a,b)$ is below ℓ. Thus, $t_1 \prec t(a,b)$. In addition, t_1 contains s'_a and s_p, thus, for any two points $a \in s'_a$ and $p \in s_p$, $t(a,p) \preceq t_1$. Therefore, $t(a,p) \prec t(a,b)$.

- $a \in s''_a$. Let t_2 be a downward triangle anchored at the intersection of $right(a,b)$ and $top(p,q)$ which has one side on the line through $right(p,q)$ (the dotted triangle in Figure 9(b)). This triangle is contained in $t(p,q)$, and has s_p on its right side. If we slide t_2 upward while its top-left corner remains on s''_a, the segment s_p remains on the right side of t_2. Thus, any triangle connecting a point $a \in s''_a$ to a point $p \in s_p$ has the same size as t_2. That is, $t(a,p) = t_2 \prec t(p,q)$.

Therefore, we have $t(a,p) \prec \max\{t(a,b), t(p,q)\}$. By similar argument we conclude that $t(b,q) \prec \max\{t(a,b), t(p,q)\}$. □

Let $\mathcal{P} = \{P_1, P_2, \dots\}$ be a partition of the points in P. Let $G(\mathcal{P})$ be the complete graph with vertex set $\mathcal{P}$. For each edge $e = (P_i, P_j)$ in $G(\mathcal{P})$, let $w(e)$ be equal to the area of the smallest triangle between a point in P_i and a point in P_j, i.e. $w(e) = \min\{t(a,b) : a \in P_i, b \in P_j\}$. That is, the weight of an edge $e \in G(\mathcal{P})$ corresponds to the size of the smallest triangle $t(e)$ defined by the endpoints of e. Let $\mathcal{T}$ be a minimum spanning tree of $G(\mathcal{P})$. Let T be the set of triangles corresponding to the edges of $\mathcal{T}$, i.e. $T = \{t(e) : e \in \mathcal{T}\}$.

Lemma 2. *The interior of any triangle in T does not contain any point of P.*

Proof. By contradiction, suppose there is a triangle $\tau \in T$ which contains a point $c \in P$. Let $e = (P_i, P_j)$ be the edge in $\mathcal{T}$ which corresponds to τ. Let a and b respectively be the points in P_i and P_j which define τ, i.e. $\tau = t(a,b)$ and $w(e) = t(a,b)$. Three cases arise: (i) $c \in P_i$, (ii) $c \in P_j$, (iii) $c \in P_l$ where $l \neq i$ and $l \neq j$. In case (i) the triangle $t(c,b)$ between $c \in P_i$ and $b \in P_j$ is smaller than $t(a,b)$; contradicts that $w(e) = t(a,b)$ in $G(\mathcal{P})$. In case (ii) the triangle $t(a,c)$ between $a \in P_i$ and $c \in P_j$ is smaller than $t(a,b)$; contradicts that $w(e) = t(a,b)$ in $G(\mathcal{P})$. In case (iii) the triangle $t(a,c)$ (resp. $t(c,b)$) between P_i and P_l (resp. P_l and P_j) is smaller than $t(a,b)$; contradicts that e is an edge in $\mathcal{T}$. □

Lemma 3. *Each point in the plane can be in the interior of at most three triangles in T.*

Proof. For each $t(a,b) \in T$, the sides $top(a,b)$, $right(a,b)$, and $left(a,b)$ contains at least one of a and b. In addition, by Lemma 2, $t(a,b)$ does not contain any point of P in its interior. Thus, none of $top(a,b)$, $right(a,b)$, and $left(a,b)$ is completely inside the other triangles. Therefore, the only possible way that two triangles $t(a,b)$ and $t(p,q)$ can share a point is that one triangle, say $t(p,q)$, contains a corner of $t(a,b)$ in such a way that a and b are outside $t(p,q)$. In other words $t(a,b)$ intersects $t(p,q)$ through one of the sides $top(p,q)$, $right(p,q)$, or $left(p,q)$. If $t(a,b)$ intersects $t(p,q)$ through a direction $d \in \{top, right, left\}$ we say that $t(p,q) \prec_d t(a,b)$.

By contradiction, suppose there is a point c in the plane which is inside four triangles $\{t_1, t_2, t_3, t_4\} \subseteq T$. Out of these four, either (i) three of them are like $t_i \prec_d t_j \prec_d t_k$ or (ii) there is a triangle t_l such that $t_l \prec_{top} t_i, t_l \prec_{right} t_j, t_l \prec_{left} t_k$, where $1 \leq i,j,k,l \leq 4$ and $i \neq j \neq k \neq l$. Figure 10 shows the two possible configurations (note that all other configurations obtained by changing the indices of triangles and/or the direction are symmetric to Figure 10(a) or Figure 10(b)).

Recall that each of t_1, t_2, t_3, t_4 corresponds to an edge in $\mathcal{T}$. In the configuration of Figure 10(a) consider t_1, t_2, and $top(t_3)$ which is shown in more detail in Figure 11(a). Suppose t_1 (resp. t_2) is defined by points a and b (resp. p and q). By Lemma 2, p and q are above $top(t_3)$, a and b are above $top(t_2)$. By Lemma 1, $\max\{t(a,p), t(b,q)\} \prec \max\{t(a,b), t(p,q)\}$. This contradicts the fact that both of the edges representing $t(a,b)$ and $t(p,q)$ are in $\mathcal{T}$, because

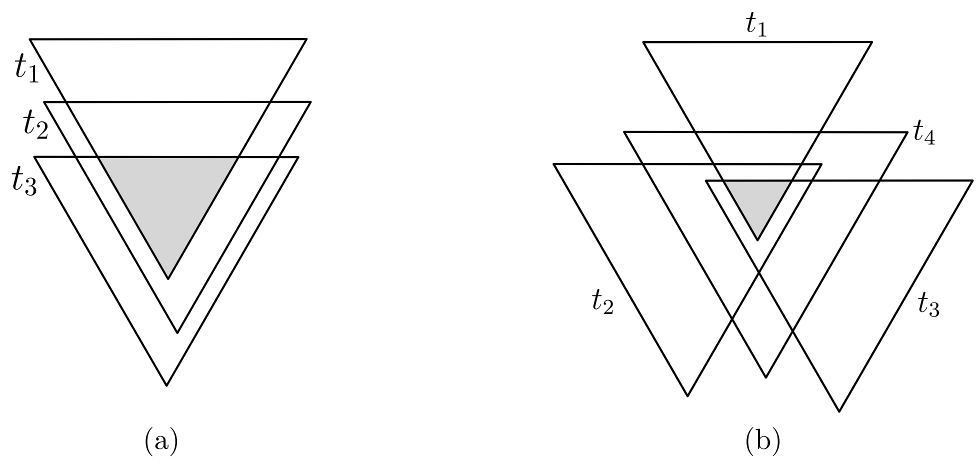


Figure 10: Two possible configurations: (a) $t_3 \prec_{top} t_2 \prec_{top} t_1$, (b) $t_4 \prec_{top} t_1, t_4 \prec_{left} t_2, t_4 \prec_{right} t_3$.

by replacing $\max\{t(a,b), t(p,q)\}$ with $t(a,p)$ or $t(b,q)$, we obtain a tree $\mathcal{T}'$ which is smaller than $\mathcal{T}$. In the configuration of Figure 10(b), consider all pairs of potential positions for two points defining t_4 which is shown in more detail in Figure 11(b). The pairs of potential positions on the boundary of t_4 are shown in red, green, and orange. Consider the red pair, and look at t_2, t_4, and $left(t_1)$. By Lemma 1 and the same reasoning as for the previous configuration, we obtain a smaller tree $\mathcal{T}'$; which contradicts the minimality of $\mathcal{T}$. By symmetry, the green and orange pairs lead to a contradiction. Therefore, all configurations are invalid; which proves the lemma.

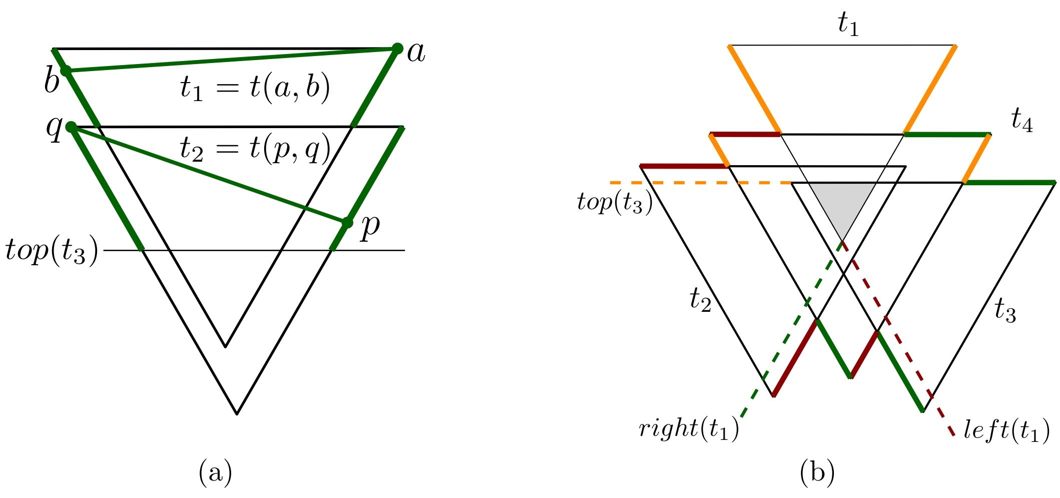


Figure 11: Illustration of Lemma 3.

□

Our results in this section are based on Lemma 2, Lemma 3, Theorem 1, and Theorem 2. Now we prove that 2-TD has a perfect matching.

Theorem 9. *For a set P of an even number of points in general position in the plane,* 2*-TD has a perfect matching.*

Proof. First we show that by removing a set K of k points from 2-TD, at most $k+1$ components are generated. Then we show that at least one of these components must be even. Finally by Theorem 1 we conclude that 2-TD has a perfect matching.

Let K be a set of k vertices removed from 2-TD, and let $\mathcal{C} = \{C_1, \dots, C_{m(k)}\}$ be the resulting $m(k)$ components, where m is a function depending on k. Actually $\mathcal{C} = $ 2-TD $- K$ and $\mathcal{P} = \{V(C_1), \dots, V(C_{m(k)})\}$ is a partition of the vertices in $P \setminus K$.

Claim 1. $m(k) \le k + 1$. Let $G(\mathcal{P})$ be the complete graph with vertex set $\mathcal{P}$ which is constructed as described above. Let $\mathcal{T}$ be a minimum spanning tree of $G(\mathcal{P})$ and let T be the set of triangles corresponding to the edges of $\mathcal{T}$. It is obvious that $\mathcal{T}$ contains $m(k) - 1$ edges and hence $|T| = m(k) - 1$. Let $F = \{(p, t) : p \in K, t \in T, p \in t\}$ be the set of all (point, triangle) pairs where $p \in K$, $t \in T$, and p is inside t. By Lemma 3 each point in K can be inside at most three triangles in T. Thus, $|F| \le 3 \cdot |K|$. Now we show that each triangle in T contains at least three points of K. Consider any triangle $\tau \in T$. Let $e = (V(C_i), V(C_j))$ be the edge of $\mathcal{T}$ which is corresponding to τ, and let $a \in V(C_i)$ and $b \in V(C_j)$ be the points defining τ. By Lemma 2, τ does not contain any point of $P \setminus K$ in its interior. Therefore, τ contains at least three points of K, because otherwise (a, b) is an edge in 2-TD which contradicts the fact that a and b belong to different components in $\mathcal{C}$. Thus, each triangle in T contains at least three points of K in its interior. That is, $3 \cdot |T| \le |F|$. Therefore, $3(m(k) - 1) \le |F| \le 3k$, and hence $m(k) \le k + 1$.

Claim 2: $o(\mathcal{C}) \le k$. By Claim 1, $|\mathcal{C}| = m(k) \le k + 1$. If $|\mathcal{C}| \le k$, then $o(\mathcal{C}) \le k$. Assume that $|\mathcal{C}| = k+1$. Since $P = K \cup \{\bigcup_{i=1}^{k+1} V(C_i)\}$, the total number of vertices of P can be defined as $n = k + \sum_{i=1}^{k+1} |V(C_i)|$. Consider two cases where (i) k is odd, (ii) k is even. In both cases if all the components in $\mathcal{C}$ are odd, then n is odd; this contradicts our assumption that P has an even number of vertices. Thus, $\mathcal{C}$ contains at least one even component, which implies that $o(\mathcal{C}) \le k$.

Finally, by Claim 2 and Theorem 1, we conclude that 2-TD has a perfect matching. □

Theorem 10. *For every set P of points in general position in the plane, 1-TD has a matching of size $\frac{2(n-1)}{5}$.*

Proof. Let K be a set of k vertices removed from 1-TD, and let $\mathcal{C} = \{C_1, \dots, C_{m(k)}\}$ be the resulting $m(k)$ components. Actually $\mathcal{C} = $ 1-TD $- K$ and $\mathcal{P} = \{V(C_1), \dots, V(C_{m(k)})\}$ is a partition of the vertices in $P \setminus K$. Note that $o(\mathcal{C}) \le m(k)$. Let M^* be a maximum matching in 1-TD. By Theorem 2,

$$|M^*| = \frac{1}{2}(n - \text{def(1-TD)}), \tag{1}$$

where

$$\begin{aligned} \text{def(1-TD)} &= \max_{K \subseteq P}(o(\mathcal{C}) - |K|) \\ &\le \max_{K \subseteq P}(|\mathcal{C}| - |K|) \\ &= \max_{0 \le k \le n}(m(k) - k). \end{aligned} \tag{2}$$

Define $G(\mathcal{P})$, $\mathcal{T}$, T, and F as in the proof of Theorem 9. By Lemma 3, $|F| \le 3 \cdot |K|$. By the same reasoning as in the proof of Theorem 9, each triangle in T has at least two points of K in its interior. Thus, $2 \cdot |T| \le |F|$. Therefore, $2(m(k) - 1) \le |F| \le 3k$, and hence

$$m(k) \le \frac{3k}{2} + 1. \tag{3}$$

In addition, $k + m(k) = |K| + |\mathcal{C}| \leq |P| = n$, and hence

$$m(k) \leq n - k. \tag{4}$$

By Inequalities (3) and (4),

$$m(k) \leq \min\{\frac{3k}{2} + 1, n - k\}. \tag{5}$$

Thus, by (2) and (5)

$$\begin{aligned} \text{def(1-TD)} &\leq \max_{0\leq k\leq n} (m(k) - k) \\ &\leq \max_{0\leq k\leq n} \{\min\{\frac{3k}{2} + 1, n - k\} - k\} \\ &= \max_{0\leq k\leq n} \{\min\{\frac{k}{2} + 1, n - 2k\}\} \\ &= \frac{n+4}{5}, \end{aligned} \tag{6}$$

where the last equation is achieved by setting $\frac{k}{2}+1$ equal to $n-2k$, which implies $k = \frac{2(n-1)}{5}$. Finally by substituting (6) in Equation (1) we have

$$|M^*| \geq \frac{2(n-1)}{5}.$$

□

6 Blocking TD-Delaunay graphs

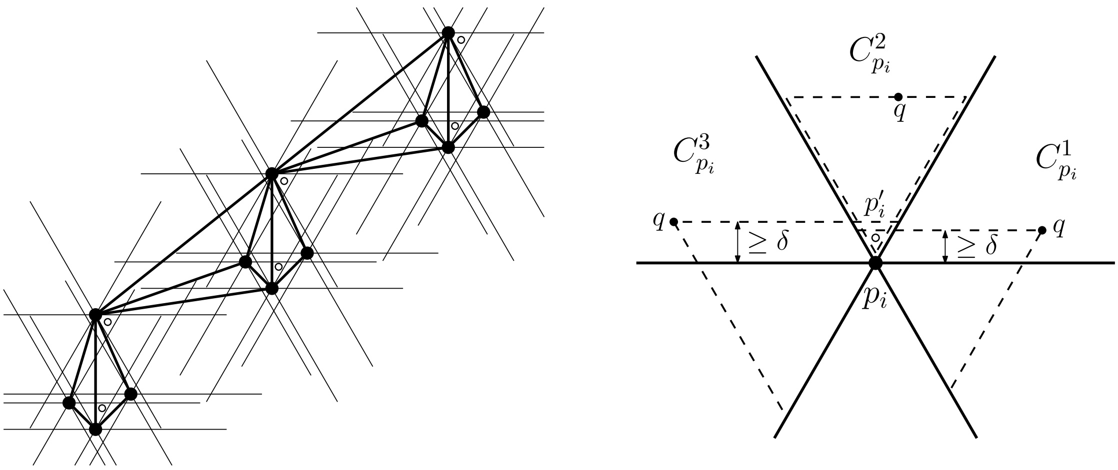
In this section we consider the problem of blocking TD-Delaunay graphs. Let P be a set of n points in the plane such that no pair of points of P is collinear in the l^0, l^{60}, and l^{120} directions. Recall that a point set K blocks k-TD(P) if in k-TD($P \cup K$) there is no edge connecting two points in P. That is, P is an independent set in k-TD($P \cup K$).

Theorem 11. *At least $\lceil\frac{(k+1)(n-1)}{3}\rceil$ points are necessary to block k-TD(P).*

Proof. Let K be a set of m points which blocks k-TD(P). Let $G(\mathcal{P})$ be the complete graph with vertex set $\mathcal{P} = P$. Let $\mathcal{T}$ be a minimum spanning tree of $G(\mathcal{P})$ and let T be the set of triangles corresponding to the edges of $\mathcal{T}$. It is obvious that $|T| = n - 1$. By Lemma 2 the triangles in T are empty, thus, the edges of $\mathcal{T}$ belong to any k-TD(P) where $k \geq 0$. To block each edge, corresponding to a triangle in T, at least $k + 1$ points are necessary. By Lemma 3 each point in K can lie in at most three triangles of T. Therefore, $m \geq \lceil\frac{(k+1)(n-1)}{3}\rceil$, which implies that at least $\lceil\frac{(k+1)(n-1)}{3}\rceil$ points are necessary to block all the edges of $\mathcal{T}$ and hence k-TD(P). □

Theorem 11 gives a lower bound on the number of points that are necessary to block a TD-Delaunay graph. By this theorem, at least $\lceil\frac{n-1}{3}\rceil$, $\lceil\frac{2(n-1)}{3}\rceil$, $n - 1$ points are necessary to block 0-, 1-, 2-TD(P) respectively. Now we introduce another formula which gives a better lower bound for 0-TD. For a point set P, let $\nu_k(P)$ and $\alpha_k(P)$ respectively denote the size of a maximum matching and a maximum independent set in k-TD(P). For every edge in the maximum matching, at most one of its endpoints can be in the maximum independent set. Thus,

$$\alpha_k(P) \leq |P| - \nu_k(P). \tag{7}$$



(a) (b)

Figure 12: (a) A 0-TD graph which is shown in bold edges is blocked by $\lceil \frac{n-1}{2} \rceil$ white points, (b) p'_i blocks all the edges connecting p_i to the vertices above $l^0_{p_i}$.

Let K be a set of m points which blocks k-TD(P). By definition there is no edge between points of P in k-TD($P \cup K$). That is, P is an independent set in k-TD($P \cup K$). Thus,

$$n \leq \alpha_k(P \cup K). \tag{8}$$

By (7) and (8) we have

$$n \leq \alpha_k(P \cup K) \leq (n+m) - \nu_k(P \cup K). \tag{9}$$

Theorem 12. *At least* $\lceil \frac{n-1}{2} \rceil$ *points are necessary to block* 0-*TD*(P).

Proof. Let K be a set of m points which blocks 0-TD(P). Consider 0-TD($P \cup K$). It is known that $\nu_0(P \cup K) \geq \lceil \frac{n+m-1}{3} \rceil$; see [4]. By Inequality (9),

$$n \leq (n+m) - \lceil \frac{n+m-1}{3} \rceil \leq \frac{2(n+m)+1}{3},$$

and consequently $m \geq \lceil \frac{n-1}{2} \rceil$ (note that m is an integer number). □

Figure 12(a) shows a 0-TD graph on a set of 12 points which is blocked by 6 points. By removing the topmost point we obtain a set with odd number of points which can be blocked by 5 points. Thus, the lower bound provided by Theorem 12 is tight.

Now let $k = 1$. By Theorem 10 we have $\nu_1(P \cup K) \geq \frac{2((n+m)-1)}{5}$, and by Inequality (9)

$$n \leq (n+m) - \frac{2((n+m)-1)}{5} = \frac{3(n+m)+2}{5},$$

and consequently $m \geq \lceil \frac{2(n-1)}{3} \rceil$; the same lower bound as in Theorem 11.

Now let $k = 2$. By Theorem 9 we have $\nu_2(P \cup K) = \lfloor \frac{n+m}{2} \rfloor$ (note that $n+m$ may be odd). By Inequality (9)

$$n \leq (n+m) - \lfloor \frac{n+m}{2} \rfloor = \lceil \frac{n+m}{2} \rceil,$$

and consequently $m \geq n$, where $n+m$ is even, and $m \geq n-1$, where $n+m$ is odd.

Theorem 13. *There exists a set* K *of* $n-1$ *points that blocks* 0-*TD*(P).

Proof. Let $d^0(p,q)$ be the Euclidean distance between l_p^0 and l_q^0. Let $\delta = \min\{d^0(p,q) : p,q \in P\}$. For each point $p \in P$ let $p(x)$ and $p(y)$ respectively denote the x and y coordinates of p in the plane. Let $p_1, \ldots, p_n$ be the points of P in the increasing order of their y-coordinate. Let $K = \{p'_i : p'_i(x) = p_i(x), p'_i(y) = p_i(y) + \epsilon, \epsilon < \delta, 1 \leq i \leq n-1\}$. See Figure 12(b). For each point p_i, let E_{p_i} (resp. $\overline{E_{p_i}}$) denote the edges of 0-TD(P) between p_i and the points above $l_{p_i}^0$ (resp. below $l_{p_i}^0$). It is easy to see that the downward triangle between p_i and any point q above $l_{p_i}^0$ (i.e. any point $q \in C_{p_i}^1 \cup C_{p_i}^2 \cup C_{p_i}^3$) contains p'_i. Thus, p'_i blocks all the edges in E_{p_i}. In addition, the edges in $\overline{E_{p_i}}$ are blocked by $p'_1, \ldots, p'_{i-1}$. Therefore, all the edges of 0-TD(P) are blocked by the $n-1$ points in K. □

We can extend the result of Theorem 13 to k-TD(P) where $k \geq 1$. For each point p_i we put $k+1$ copies of p'_i very close to p_i. Thus,

Theorem 14. *There exists a set K of $(k+1)(n-1)$ points that blocks k-TD(P).*

This bound is tight. Consider the case where $k = 0$. In this case 0-TD(P) can be a path representing $n-1$ disjoint triangles and for each triangle we need at least one point to block its corresponding edge.

7 Conclusion

In this paper, we considered some combinatorial properties of higher-order triangular-distance Delaunay graphs of a point set P. We proved that

- k-TD is $(k+1)$ connected.
- 1-TD contains a bottleneck biconnected spanning graph of P.
- 7-TD contains a bottleneck Hamiltonian cycle and 5-TD may not have any.
- 6-TD contains a bottleneck perfect matching and 5-TD may not have any.
- 1-TD has a matching of size at least $\frac{2(n-1)}{5}$.
- 2-TD has a perfect matching when P has an even number of points.
- $\lceil \frac{n-1}{2} \rceil$ points are necessary to block 0-TD.
- $\lceil \frac{(k+1)(n-1)}{3} \rceil$ points are necessary and $(k+1)(n-1)$ points are sufficient to block k-TD.

We leave a number of open problems:

- What is a tight lower bound for the size of maximum matching in 1-TD?
- Does 6-TD contain a bottleneck Hamiltonian cycle?
- As shown in Figure 1(a) 0-TD may not have a Hamiltonian cycle. For which values of $k = 1, \ldots, 6$, is the graph k-TD Hamiltonian?

References

[1] M. Abellanas, P. Bose, J. García-López, F. Hurtado, C. M. Nicolás, and P. Ramos. On structural and graph theoretic properties of higher order Delaunay graphs. *Int. J. Comput. Geometry Appl.*, 19(6):595–615, 2009.

[2] O. Aichholzer, R. F. Monroy, T. Hackl, M. J. van Kreveld, A. Pilz, P. Ramos, and B. Vogtenhuber. Blocking Delaunay triangulations. *Comput. Geom.*, 46(2):154–159, 2013.

[3] B. Aronov, M. Dulieu, and F. Hurtado. Witness Gabriel graphs. *Comput. Geom.*, 46(7):894–908, 2013.

[4] J. Babu, A. Biniaz, A. Maheshwari, and M. Smid. Fixed-orientation equilateral triangle matching of point sets. To appear in *Theoretical Computer Science.*

[5] C. Berge. Sur le couplage maximum d'un graphe. *C. R. Acad. Sci. Paris*, 247:258–259, 1958.

[6] N. Bonichon, C. Gavoille, N. Hanusse, and D. Ilcinkas. Connections between theta-graphs, Delaunay triangulations, and orthogonal surfaces. In *WG*, pages 266–278, 2010.

[7] P. Bose, P. Carmi, S. Collette, and M. H. M. Smid. On the stretch factor of convex Delaunay graphs. *Journal of Computational Geometry*, 1(1):41–56, 2010.

[8] P. Bose, S. Collette, F. Hurtado, M. Korman, S. Langerman, V. Sacristan, and M. Saumell. Some properties of k-Delaunay and k-Gabriel graphs. *Comput. Geom.*, 46(2):131–139, 2013.

[9] M.-S. Chang, C. Y. Tang, and R. C. T. Lee. 20-relative neighborhood graphs are Hamiltonian. *Journal of Graph Theory*, 15(5):543–557, 1991.

[10] M.-S. Chang, C. Y. Tang, and R. C. T. Lee. Solving the Euclidean bottleneck biconnected edge subgraph problem by 2-relative neighborhood graphs. *Discrete Applied Mathematics*, 39(1):1–12, 1992.

[11] M.-S. Chang, C. Y. Tang, and R. C. T. Lee. Solving the Euclidean bottleneck matching problem by k-relative neighborhood graphs. *Algorithmica*, 8(3):177–194, 1992.

[12] P. Chew. There are planar graphs almost as good as the complete graph. *J. Comput. Syst. Sci.*, 39(2):205–219, 1989.

[13] M. B. Dillencourt. A non-hamiltonian, nondegenerate Delaunay triangulation. *Inf. Process. Lett.*, 25(3):149–151, 1987.

[14] M. B. Dillencourt. Toughness and Delaunay triangulations. *Discrete & Computational Geometry*, 5:575–601, 1990.

[15] H. N. Gabow and R. E. Tarjan. Algorithms for two bottleneck optimization problems. *J. Algorithms*, 9(3):411–417, 1988.

[16] T. Kaiser, M. Saumell, and N. V. Cleemput. 10-Gabriel graphs are Hamiltonian. *To appear in Information Processing Letters*, 2015.

[17] T. Lukovszki. New results of fault tolerant geometric spanners. In *WADS*, pages 193–204, 1999.

[18] W. T. Tutte. The factorization of linear graphs. *Journal of the London Mathematical Society*, 22(2):107–111, 1947.