MULTI-ACTION L.R.P. SCHEMES

$$p_i(n+1) = k_1 p_i \quad \text{if} \quad \alpha_j \text{ is chosen}, \beta = 0$$

$$= 1 - k_1 \sum_{j \neq i}' p_j \quad \alpha_i \text{ is chosen} \quad \beta = 0$$

$$= k_2 p_i \quad \alpha_i \text{ is chosen} \quad \beta = 1$$

$$= k_2 p_i + \frac{1 - k_2}{R-1} \quad \alpha_j \text{ is chosen}, \quad \beta = 1.$$

$\therefore p_i(n+1)$ is a random variable.

$p_i(n+1) \big| \underline{p}$ has the following distribution

$$p_i(n+1) = k_1 p_i \qquad \text{w.p.} \qquad \sum_{j \neq i} p_j (1 - c_j)$$

$$= 1 - k_1 \sum_{j \neq i}' p_j \qquad p_i (1 - c_i)$$

$$= k_2 p_i \qquad p_i c_i$$

$$= k_2 p_i + \frac{1 - k_2}{R-1} \qquad \sum_{j \neq i}' p_j c_j$$

$$E\left[\, p_i(n+1) \mid \underline{p} \,\right]$$

$$= \cancel{k_1 p_i \sum_{j\neq i}' p_j} \quad - \quad k_1 p_i \sum_{j \neq i}' p_j c_j \quad + \quad p_i (1 - c_i)$$

$$- \cancel{k_1 p_i \sum_{j\neq i}' p_j} \quad + \quad k_1 p_i c_i \sum_{j \neq i} p_j$$

$$+ \; k_2 p_i^2 c_i \quad + \; k_2 p_i \sum_{j \neq i} p_j c_j \; + \; \frac{1-k_2}{R-1} \sum_{j \neq i}' p_j c_j$$

$$= (k_2 - k_1) p_i \sum_{j \neq i}' p_j c_j + (k_2 - k_1) c_i p_i^2 + p_i (1 - c_i + k_1 c_i)$$

$$+ \; \frac{1-k_2}{R-1} \sum_{j \neq i}' p_j c_j$$

$$= (k_2 - k_1) p_i \sum_{j=1}^{R}{}' p_j c_j \; + \; p_i (1 - c_i + k_1 c_i) \; + \; \frac{1-k_2}{R} \sum_{j \neq i}' p_j c_j$$

QUADRATIC in $p_j p_i$, (in $\underline{p}$)

except if $k_1 = k_2$

i.e. SYMMETRIC $L_{R,P}$ SCHEME

<u>IF $k_1 = k_2 = k$</u>

$$E\left[p_i(n+1)\right] = \left(1 - c_i + k c_i\right) E\left[p_i\right] + \frac{1-k}{R-1} \sum_{j \neq i}' c_j \, E\left[p_j\right]$$

$E\left[\underline{p}\right]$ is again a Markov chain.

$$E\left[\underline{p}(n+1)\right] = \begin{bmatrix} 1-c_1+kc_1 & \frac{(1-k)c_1}{R-1} & \cdots & \frac{(1-k)c_1}{R-1} \\ \frac{(1-k)c_2}{R-1} & 1-(1-k)c_2 & \cdots & \frac{(1-k)c_2}{R-1} \\ \vdots & & & \vdots \\ \frac{(1-k)c_R}{R-1} & \cdots & & 1-(1-k)c_R \end{bmatrix}^T E\left[\underline{p}(n)\right]$$

$$E\left[\underline{p}(n+1)\right] = A^T E\left[\underline{p}(n)\right]$$

A is stochastic matrix.

Let $1-k = e$.

$$A = \begin{bmatrix} 1-ec_1 & \frac{ec_1}{R-1} & \cdots & \frac{ec_1}{R-1} \\ \frac{ec_2}{R-1} & 1-ec_2 & \cdots & \frac{ec_2}{R-1} \\ & & & \\ \frac{ec_R}{R-1} & \frac{ec_R}{R-1} & \cdots & 1-ec_R \end{bmatrix}$$

To solve for limiting vector ..

$$\underline{p}^* = A^T \underline{p}^*$$

i.e. $\left[I - A \right]^T \underline{p}^* = 0$

$$\begin{bmatrix} ec_1 & -\frac{ec_2}{R-1} & \cdots & -\frac{ec_K}{R-1} \\ -\frac{ec_1}{R-1} & ec_2 & \cdots & -\frac{ec_R}{R-1} \\ \vdots & & & \\ -\frac{ec_1}{R-1} & & \cdots & \frac{ec_R}{R-1} \end{bmatrix} \begin{bmatrix} p_1^* \\ p_2^* \\ \vdots \\ p_R^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Expand first row: $\qquad (e \neq 0 \text{ i.e. } k = 1)$

$$e\left[c_1 p_1^* - \frac{1}{R-1}\sum_{i=2}^{R} c_i p_i^*\right] = 0$$

$$\Rightarrow \quad c_1 p_1^* - \frac{1}{R-1}\sum_{i=2}^{R} c_i p_i^*$$

$$c_1 p_1^* + \frac{1}{R-1} c_1 p_1^* - \frac{1}{R-1}\sum_{i=1}^{R} c_i p_i^* = 0$$

$$c_1 p_1^*\left[1 + \frac{1}{R-1}\right] = \frac{1}{R-1}\xi$$

where $\xi = \sum_{i=1}^{R} c_i p_i^*$, ind. of i.

$$c_1 p_1^* \frac{R}{R-1} = \frac{1}{R-1}\xi$$

i.e. $\quad p_1^* = \dfrac{1}{R c_1}\xi.$

llly. $\quad p_2^* = \dfrac{1}{R c_2}\xi.$

$\vdots$

$\quad p_R^* = \dfrac{1}{R c_R}\xi$

Add. $\quad 1 = \dfrac{\xi}{R}\left(\dfrac{1}{c_1} + \dfrac{1}{c_2} + \cdots \dfrac{1}{c_R}\right)$

Hence

$$\xi = \frac{R}{\sum_{j=1}^{R} 1/c_j}$$

$// \quad \frac{c_2}{c_1+c_2}$

$$\boxed{\therefore \; p_i^* = \frac{\xi}{R c_i} = \frac{\frac{1}{c_i}}{\sum_{j=1}^{R} \frac{1}{c_j}}}$$

inv. prop. to c_i

if $c_i < c_j$
$p_i^* > p_j$

What is M^*

$$M^* = \sum_i c_i p_i^* = \frac{R}{\sum_{j=1}^{R} \frac{1}{c_j}} = H = \text{Harmonic Mean}$$

$$M(0) = \frac{1}{R} \sum_{i=1}^{R} c_i = A = \text{Arithmetic Mean.}$$

Since $H < A$

THE $L_{R,P}$ scheme is Expedient.

① IND. OF $P(0)$
② IND. OF ENVT.
ind. of 'k'

Note : SPEED & VARIANCE CONTROLLED BY 'k'.